Intermediate extension of Chow motives of Abelian type

by

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Abstract

In this article, we give an unconditional construction of a motivic analogue of the intermediate extension in the context of Chow motives of Abelian type. Our main application concerns intermediate extensions of Chow motives associated to Kuga families to the Baily–Borel compactification of a Shimura variety .

Keywords: weight structures, semi-primary categories, Chow motives, motivic intermediate extension, Shimura varieties.

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Contents

0 Introduction

A profound conjecture concerning the category of motives over some base X predicts the existence of a t-structure, all of whose realizations are compatible with the so-called *perverse* t-structure. This structure would in particular allow for the construction of the *intermediate extension* to X of any Chow motive over an open sub-scheme U of X, canonically up to (unipotent) automorphisms restricting to the identity on U.

This latter construction is carried out in [CH] for quasi-projective varieties X over $\mathbb{C}$, assuming Grothendieck's standard conjectures and Murre's filtration conjecture. Note that for X equal to the spectrum of a field of charcteristic zero, these two conjectures would be implied by the existence of a t-structure [Be].

The aim of this paper is to establish the theory of intermediate extensions. Its ingredients are radically different from t-structures. It is based on two key notions: *weight structures* à la Bondarko, and *semi-primary categories* à la André–Kahn. This approach is *formula free*. Above all, it is *unconditional*, once certain geometric conditions are satisfied.

In order to illustrate our results, let us discuss their implications in a context which lends itself to important arithmetic applications. Let M^L be a

pure Shimura variety, and assume that the group L to which it is associated, is *neat*, which implies that M^L is smooth over the *reflex field* E. The variety M^L is the target of proper, smooth morphisms

$$\pi : M^K \longrightarrow M^L ,$$

induced by a morphism of *Shimura data* $(P, \mathfrak{X}) \to (G, \mathfrak{H})$, which identifies G with the maximal reductive quotient of P. We make a mild technical assumption on the Shimura data $(G, \mathfrak{H})$, namely that they satisfy [P2, Condition (3.1.5)] (this condition will be recalled later). The source of π is a *Kuga variety*, *i.e.*, a *mixed Shimura variety* admitting the structure of a torsor under an Abelian scheme over M^L. (Note that we admit the case $\pi = \mathrm{id}_{M^L}$, *i.e.*, the Abelian scheme may be of relative dimension zero.) Fix one such π. The scheme M^L being regular, and π proper and smooth, the motive

$$\pi_* \mathbb{1}_{M^K}$$

belongs to the category $CHM^s(M^L)_{\mathbb{Q}}$ of *smooth Chow motives* over M^L [Le2]. Fix an extension F of $\mathbb{Q}$, and a direct factor N of $\pi_* \mathbb{1}_{M^K}$, viewed as an object of the category $CHM(M^L)_F$ of *Chow motives.* Denote by $j : M^L \hookrightarrow (M^L)^*$ the *Baily–Borel compactification* [AMRT, BB], and by $i : \partial(M^L)^* \hookrightarrow (M^L)^*$ the closed immersion of the complement of M^L.

Theorem 0.1. *(a) There is a Chow motive $j_{!*} N \in CHM((M^L)^*)_F$ extending N,* i.e.*, $j^* j_{!*} N \cong N$, and satisfying the following properties.*

(1) $j_{!*} N$ admits no non-zero direct factor belonging to $i_* CHM(\partial(M^L)^*)_F$.

(2) Any element of the kernel of

$$j^* : \mathrm{End}_{CHM((M^L)^*)_F}(j_{!*} N) \longrightarrow \mathrm{End}_{CHM(M^L)_F}(N)$$

is nilpotent.

(b) Among the extensions of N to a Chow motive over $(M^L)^*$, $j_{!*} N$ is characterized up to isomorphism by each of the properties (1) and (2).

(c) Any extension of N to a Chow motive over $(M^L)^*$ is isomorphic to a direct sum $j_{!*} N \oplus i_* N_\partial$, for an object N_∂ of $CHM(\partial(M^L)^*)_F$.

(d) Let f be an idempotent endomorphism of N. Then f admits an idempotent extension to $j_{!*} N$.

For certain Shimura varieties, parts of Theorem 0.1 are known. This concerns modular curves [Sll], Hilbert–Blumenthal varieties [GHM], and Shimura varieties of dimension 3 [MS]; note that in the latter two cases the results are obtained only after a base change from the reflex field E to the field $\mathbb{C}$ of complex numbers. Over E itself, apart from [Sll], there are the rather recent results from [V] and [NV2] concerning Hilbert–Blumenthal varieties and Siegel threefolds. We refer to Remark 8.10 for details.

Note that the restriction j^* is full on Chow motives [W4, Thm. 1.7 (b)]. Thus, the isomorphisms in (b) and (c) can be chosen to give the identity on N, when restricted to M^L. The ambiguity related to the presence of a kernel of j^* can be controlled by passing to a suitable quotient category of $CHM((M^L)^*)_F$. This point of view will be developed in Section 2; it also allows to treat functoriality in a satisfactory manner. We refer to $j_{!*}N$ as the *intermediate extension* of N. This terminology is justified thanks to our next result.

Theorem 0.2. *The intermediate extension is compatible with the Betti realization [Ay2]. More precisely, denote by*

$$R(N) \in D^b_c\bigl(M^L(\mathbb{C}), F\bigr) \quad \textit{and} \quad R(j_{!*}N) \in D^b_c\bigl((M^L)^*(\mathbb{C}), F\bigr)$$

the images of N and $j_{!*}N$ under the Betti realization, and by

$$H^n : D^b_c(\bullet, F) \longrightarrow \mathbf{Perv}_c(\bullet, F)\ ,\ n \in \mathbb{Z}$$

the perverse cohomology functors. Then for any integer n, there is a canonical and functorial isomorphism

$$H^n R(j_{!*}N) \cong j_{!*} H^n R(N)$$

of perverse sheaves on $(M^L)^*(\mathbb{C})$.

Denote by m the structure morphism $(M^L)^* \to \mathbf{Spec}\, E$. From the compatibility of the Betti realization with direct images [Ay2], we deduce the following.

Corollary 0.3. *Assume that $R(N)$ is concentrated in a single perverse degree d. Then $R(j_{!*}N)$ is concentrated in perverse degree d, and the complex computing (singular)* intersection cohomology *of $(M^L)^*(\mathbb{C})$ with coefficients in $R(N)$ is of motivic origin. More precisely, there is a canonical isomorphism*

$$R(m_* j_{!*}N) \cong m_* j_{!*} R(N)\ .$$

According to [DM, Thm. 3.1], the motive $\pi_* \mathbb{1}_{M^K}$ admits a Chow–Künneth decomposition. Any direct factor N contained in a Chow–Künneth component then satisfies the additional hypothesis of Corollary 0.3. Given its statement, it appears justified to think of the object $m_* j_{!*}N$ as being the *intersection motive* of M^L with coefficients in N (with respect to the Baily–Borel compactification). Note that for certain Shimura varieties, the construction of the intersection motive, or at least partial information entering that construction, already appears in the literature. To the cases which result from the application of m_* to the intermediate extension [Sll, GHM, MS, V, NV2], one needs to add surfaces with constant coefficients [CM] and Hilbert–Blumenthal varieties with non-constant coefficients [W3]. Again, we refer to Remark 8.10 for more details.

Now let $[\,\cdot h] : M^L \to M^{L'}$ be a finite morphism associated to change of the "level" L. It extends to a finite morphism $(M^L)^* \to (M^{L'})^*$, denoted by the same symbol $[\,\cdot h]$. Assume that L' is neat, too.

Theorem 0.4. *The intermediate extension is compatible with* $[\,\cdot h]_*$. *More precisely, the Chow motive* $[\,\cdot h]_* \, j_{!*} N \in CHM((M^{L'})^*)_F$ *satisfies the analogues of the properties (1), (2) from Theorem 0.1 (a). Furthermore, the analogues of Theorem 0.1 (b)–(d) hold for* $[\,\cdot h]_* \, j_{!*} N$ *and* $[\,\cdot h]_* N$ *instead of* $j_{!*} N$ *and* N.

Finally, let us discuss *Hecke operators*. In order to do so, we need to fix a section i of the projection π from P to G, and to suppose that the group K contains the image of L under i. In other words, K is the semi-direct product of L and the kernel of the restriction of π to K. Fix $x \in G(\mathbb{A}_f)$, and consider the double coset LxL, which is one of the generators of the *Hecke algebra* $R(L, G(\mathbb{A}_f))$ associated to M^L. The above results will be shown to imply the following.

Theorem 0.5. *(a) The Hecke operator* LxL *acts on* $m_* \, j_{!*} N$, *in a way compatible with its action on* $m_* j_! N$ *and on* $m_* j_* N$.
(b) Assume that the Betti realization $R(N)$ *is concentrated in a single perverse degree. Then the endomorphism*

$$R(LxL) : R(m_* \, j_{!*} N) \longrightarrow R(m_* \, j_{!*} N)$$

coincides with the Hecke operator defined on the complex computing intersection cohomology via *the isomorphism*

$$R(m_* \, j_{!*} N) \cong m_* \, j_{!*} \, R(N)$$

from Corollary 0.3.

Using Faltings-Chai's results on *integral models* of Baily–Borel and toroidal compactifications of Siegel varieties M^L, and on integral models of compactifications of Kuga families over M^L [FC], one can show that analogues of Theorems 0.1–0.5 hold for Chow motives occurring as direct factors in the relative motive of the integral models of Kuga families (Theorems 8.12–8.14). Given Lan's recent generalization [La1, La2], it appears likely that such analogues of Theorems 0.1–0.5 actually exist for integral models of arbitrary *PEL-type Shimura varieties*. We refer to Remark 8.11 for more precise comments.

The above results will be shown to result from a more general formalism. Here are the rough ideas: (1) apply the generalization of Kimura's notion of *finite dimensionality* [AK, Déf. 9.1.1] to the category of smooth Chow motives over a base, (2) *glue* certain finite dimensional Chow motives over strata of the base, and study the finiteness conditions which are respected by the gluing process, (3) show that the finiteness conditions in question are strong

enough to allow for a formulation of a theory of intermediate extensions. It turns out that finite dimensionality itself is in general not preserved by gluing. This has an obvious reason: in general, gluing will not preserve rigidity (basically because it will not preserve smoothness). One of the main results from [AK] ([loc. cit.], Thm. 9.2.2; see also [O'S1, Lemma 4.1]) states that a rigid category, all of whose objects are finite dimensional, is semi-primary. It turns out that semi-primality glues well. Actually, the only abstract ingredient one then needs in order to formulate a theory of intermediate extension, is the weight structure on motives, identifying Chow motives as its *heart*.

Let us now give a more detailed description of the content of this paper. Sections 1 and 2 can be read independently of the rest of this article, since they are of purely homological nature. From the beginning, we impose the finiteness condition which turns out to be be "right" one, and thus concentrate exclusively on semi-primary categories, *i.e.*, additive categories $\mathfrak{A}$ which are semi-simple up to "nilpotency phenomena", encoded in a condition on the *radical* of $\mathfrak{A}$. Our main homological tool can be obtained directly from the definitions: according to Theorem 1.3, any morphism in a pseudo-Abelian semi-primary category $\mathfrak{A}$ is a direct sum of an isomorphism and a morphism belonging to the radical of $\mathfrak{A}$. The proof of Theorem 1.3 is a variation on a classical theme: elements of a ring A which are idempotent modulo a nil-ideal $\mathfrak{n}$, can be modified by an element of $\mathfrak{n}$ such that the result is idempotent in A [R, 1.1.28]. We then turn to a relative situation, and study a full inclusion $i_* : \mathfrak{B} \hookrightarrow \mathfrak{A}$ of one additive category into another. We anticipate one of the defining features of the intermediate extension, and therefore consider morphisms in $\mathfrak{A}$ between objects of $\mathfrak{B}$ on the one hand, and objects A admitting no non-zero direct factor belonging to $\mathfrak{B}$ on the other. In that context, Theorem 1.3 is applied a first time. Its Corollary 1.5 states that if $\mathfrak{B}$ is semi-primary, then all such morphisms are in the radical; furthermore, A can be split off any object of $\mathfrak{A}$ having the same image as A in the quotient category $\mathfrak{A}/\mathfrak{B}$. Modulo a mild technical assumption, satisfied in the geometric context we shall consider later (and which will therefore be ignored for the purpose of this introduction), this observation leads to a first approximation of the intermediate extension: according to Theorem 1.9, the quotient morphism $j^* : \mathfrak{A} \to \mathfrak{A}/\mathfrak{B}$ admits a partial right inverse. More precisely, a sub-ideal $\mathfrak{g}$ of the radical of $\mathfrak{A}$ can be identified such that j^* factors through $\mathfrak{A}/\mathfrak{g}$, and such that the factorization admits a canonical left inverse; it is that left inverse that will be defined as the intermediate extension. Let us note that our requirement for the categories to be pseudo-Abelian could sometimes be dropped, provided that direct factors be replaced by retracts. We suppose the coefficient ring F of the semi-primary categories to be a finite direct product of fields in order to have the results from [AK] at our disposal.

From Section 2 onwards, weight structures enter the picture. The considerations of Section 1 are applied to the *heart* of a weight structure. In

this context, Theorem 1.3 is used a second time: according to Theorem 2.2, semi-primality of the heart implies the existence of *minimal weight filtrations*. Following what we consider as our main insight, the intermediate extension of an object on an open sub-scheme U of a scheme X should correspond to the minimal weight filtration on the "boundary" object on $X - U$. This insight is quantified in Theorem 2.9, where we consider an abstract gluing situation "$X = U \coprod Z$", equipped with weight structures. Applying Theorem 1.9, we see that the intermediate extension exists once we impose semi-primality over Z. Furthermore, we show that semi-primality glues: if it holds over Z and over U, then it holds over X. This latter result opens the way to inductive constructions of weight structures with semi-primary hearts, as they will be performed later on. Summary 2.12 then contains all properties of the abstract intermediate extension which we shall apply in subsequent chapters.

Although all results in the present work are unconditional, it appeared useful to insert a section containing two conjectures guiding our vision, as far as motives are concerned. Section 3 starts off with a short review of weight structures on *constructible Beilinson motives* over a base X [CD]. In particular, we recall the main results from [H], assuring the existence of such structures. We also recall the definition of the category $CHM(X)$ of Chow motives over X as the heart of the motivic weight structure on the category of Beilinson motives over X [W4]. Conjectures 3.3 and 3.4 then affirm that $CHM(X)$ is semi-primary, and that indecomposable Chow motives over X have trivial radical. The first of these statements should be viewed as a "construcible" analogue in the relative context of a conjecture proposed independently by Kimura and O'Sullivan, predicting that over a field, all Chow motives are finite dimensional. Given the theory that was developed in Sections 1 and 2, it would imply the existence of the intermediate extension for any base X.

The aim of Sections 4 and 5 is to establish semi-primality for certain explicit sub-categories of Chow motives. In order to do so, we first need to study the behaviour of motives under gluing. The geometric context we study from now on is therefore that of a stratified scheme Y, and of stratified morphisms. A first non-trivial example that comes to mind concerns Tate motives [Le1, Le3]. Theorem 4.4 gives a sufficient criterion for the categories of Tate motives over the strata Y_φ to glue: this is the case as soon as the closures of the strata are regular. The main ingredient of the proof is *absolute purity* [CD, Thm. 14.4.1]. Let us insist that no additional conditions on the relative situation of the strata (*e.g.*, transversality) are needed. For the sequel, it will be important to generalize Theorem 4.4 to relative situations, where we consider direct images of Tate motives under proper morphisms. This is the content of Corollary 4.10. In each of these settings, the gluing is shown to respect the weight structures; in particular, the glued categories are all generated by their hearts.

In Section 5, we put everything together, and formulate our Main Theorem 5.4. Following the rough idea sketched further above, we aim at gluing sub-categories of motives generated by finite dimensional Chow motives over the strata Y_φ, to obtain a semi-primary category of Chow motives over Y. Given that in pratice, a proper morphism π over a stratified scheme Y rarely admits a nice description over the individual strata Y_φ, unless the pre-images of the strata are stratified further, we need the assumptions on the induced morphisms to be modified: basically, we sacrifice properness of the strata of π in order to obtain regularity of the strata of the source. Since Tate twists do not affect finite dimensionality, we obtain a condition on the strata of π: they factorize over schemes B whose motive is finite dimensional over Y_φ, and such that the strata of the source over B give rise to Tate motives. According to the results proved before, we get a weight structure on the glued category, whose heart is semi-primary. The intermediate extension therefore exists; its main properties are recalled in Corollary 5.7. We conclude Section 5 with a list of examples, showing in particular that our theory in non-empty. In particular (Example 5.9 (b), (c)), a recent generalization of [Kü, Thm. (3.3.1)] (see [O'S2, pp. 54–55, lower half of p. 61]) implies that finite-dimensionality is satisfied whenever B is a torsor under an Abelian scheme over Y_φ.

Sections 6 and 7 contain our main compatibility results concerning the intermediate extension. Theorems 6.6 and 6.7 are about direct images under certain finite morphisms, and inverse images under certain smooth morphisms, respectively. Theorem 7.2 states that the intermediate extension is compatible with the Betti realization [Ay2]. It shows in particular that the approach *via* weight structures is indeed the "right" one in the motivic context. All three proofs have one point in common: the proof that the respective functor in question (direct image, inverse image, realization) is *radicial* [AK, Déf. 1.4.6], *i.e.*, maps the radical of the source to the radical of the target. In the context of Section 6, this results from the study of the situation over strata — here, we were lucky enough to have strong results from [O'S2] at our disposal — and a rather general abstract principle: "radiciality is compatible with gluing" (we refer to Proposition 2.17 for a precise statement). In principle, the same strategy is used in the proof of Theorem 7.2. However, already over strata we are confronted with a rather serious obstacle: it turns out that radiciality of the realization is directly related to the conjecture "numerical equivalence equals (Betti) homological equivalence". This explains why Theorem 7.2 requires our most restrictive hypothesis: finite dimensionality alone does not seem to formally imply the conjecture, we need to invoke Lieberman's result on the validity of the conjecture for Abelian varieties (Theorem 7.12). Once the situation over strata is settled, there is another problem, this time related to gluing. The principle sketched above (Proposition 2.17) is valid only if both the source and the target of the functor in question carry a weight structure. But this is not the case for the

bounded derived category of sheaves. In a sense, radiciality of the realization would thus be easier to prove if the Hodge theoretical realization were known to exist. Given that at present, this is not the case, the statement and the proof of our relevant gluing result (Corollary 7.11) is somewhat less elegant than what one could hope.

In Section 8, we show how to deduce the results stated further above from the general theory developed in the preceding sections.

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Conventions: Throughout the article, F denotes a finite direct product of fields (which are supposed to be of characteristic zero from Section 3 onwards). We fix a base scheme $\mathbb{B}$, which is of finite type over some excellent scheme of dimension at most two. By definition, *schemes* are $\mathbb{B}$-schemes which are separated and of finite type (in particular, they are excellent, and Noetherian of finite dimension), *morphisms* between schemes are separated morphisms of $\mathbb{B}$-schemes, and a scheme is *regular* if the underlying reduced scheme is regular in the usual sense.

1 Gluing of semi-primary categories

Let us fix our coefficients F, and consider a category $\mathfrak{A}$ which is F-linear [AK, Sect. 1.1]. Recall the following definitions.

Definition 1.1 ([Ke])**.** The *radical* of $\mathfrak{A}$ is the ideal $\mathrm{rad}_{\mathfrak{A}}$ which associates to each pair of objects A, B of $\mathfrak{A}$ the subset

$$\mathrm{rad}_{\mathfrak{A}}(A,B) := \{f \in \mathrm{Hom}_{\mathfrak{A}}(A,B)\ ,\ \forall\, g \in \mathrm{Hom}_{\mathfrak{A}}(B,A)\ ,\ \mathrm{id}_A - gf \text{ invertible}\}$$

of $\mathrm{Hom}_{\mathfrak{A}}(A,B)$.

In [AK, Déf. 1.4.1], the radical is referred to as the *Kelly radical* of $\mathfrak{A}$. It can be checked that $\mathrm{rad}_{\mathfrak{A}}$ is indeed a two-sided ideal of $\mathfrak{A}$ in the sense of [AK, Sect. 1.3], *i.e.*, for each pair of objects A, B, $\mathrm{rad}_{\mathfrak{A}}(A,B)$ is an F-submodule of $\mathrm{Hom}_{\mathfrak{A}}(A,B)$, and for each pair of morphisms $h : A' \to A$ and $k : B \to B'$ in $\mathfrak{A}$,

$$k\,\mathrm{rad}_{\mathfrak{A}}(A,B)h \subset \mathrm{rad}_{\mathfrak{A}}(A',B')\ .$$

By definition (*cf.* [AK, first statement of Prop. 1.4.4 b)]), $\mathrm{rad}_{\mathfrak{A}}$ is maximal among the two-sided ideals $\mathfrak{I}$ of $\mathfrak{A}$ such that the projection onto the quotient category

$$\mathfrak{A} \longrightarrow\!\!\!\!\!\rightarrow \mathfrak{A}/\mathfrak{I}$$

is conservative.

Definition 1.2 ([AK, Déf. 2.3.1])**.** The F-linear category $\mathfrak{A}$ is called *semi-primary* if
(1) for all objects A of $\mathfrak{A}$, the radical $\mathrm{rad}_{\mathfrak{A}}(A, A)$ is nilpotent, *i.e.*, there exists a positive integer N such that

$$\mathrm{rad}_{\mathfrak{A}}(A, A)^N = 0 \subset \mathrm{End}_{\mathfrak{A}}(A) \ ,$$

(2) the F-linear quotient category

$$\overline{\mathfrak{A}} := \mathfrak{A}/\,\mathrm{rad}_{\mathfrak{A}}$$

is semi-simple.

Among the basic properties of semi-primary categories, let us mention one which we shall frequently employ: according to [AK, Prop. 2.3.4 b)], the category $\overline{\mathfrak{A}}$ is Abelian if (and only if) $\mathfrak{A}$ is pseudo-Abelian.

Here is our main homological tool; it is the key for everything to follow.

Theorem 1.3. *Assume $\mathfrak{A}$ to be a pseudo-Abelian semi-primary F-linear category. Let*

$$f : A \longrightarrow B$$

be a morphism in $\mathfrak{A}$. Then there exist decompositions

$$A = A^r \oplus A^s \ , \ B = B^r \oplus B^s \ ,$$

such that
(a) the decompositions are respected by f:

$$f = f^r \oplus f^s \in \mathrm{Hom}_{\mathfrak{A}}(A^r, B^r) \oplus \mathrm{Hom}_{\mathfrak{A}}(A^s, B^s) \subset \mathrm{Hom}_{\mathfrak{A}}(A, B) \ ,$$

(b) the morphism f^r belongs to the radical $\mathrm{rad}_{\mathfrak{A}}(A^r, B^r)$*,*
(c) the morphism f^s is an isomorphism.
The isomorphism classes of A^r, A^s, B^r and B^s are uniquely determined by properties (a)–(c).

Proof. By [AK, Prop. 2.3.4 b)], the category $\overline{\mathfrak{A}}$ is Abelian. Therefore, the morphism

$$\bar{f} := f \mod \mathrm{rad}_{\mathfrak{A}}$$

admits both a kernel and an image. In addition, $\overline{\mathfrak{A}}$ is semi-simple, hence direct complements $\bar{A}^s$ to $\ker(\bar{f})$ and $\bar{B}^r$ to $\mathrm{im}(\bar{f})$ can be chosen:

$$A = \ker(\bar{f}) \oplus \bar{A}^s \;,\; B = \bar{B}^r \oplus \mathrm{im}(\bar{f}) \quad \text{in} \quad \overline{\mathfrak{A}} \;.$$

Fix such choices, and denote by $\bar{g}$ the associated projection of B onto $\mathrm{im}(\bar{f})$, followed by the inverse of the restriction of $\bar{f}$ to $\bar{A}^s$. Thus, both $\bar{f}\bar{g}$ and $\bar{g}\bar{f}$ are idempotent; actually, we even have the relations

$$\bar{f} = \bar{f}\bar{g}\bar{f} \quad \text{and} \quad \bar{g} = \bar{g}\bar{f}\bar{g} \;.$$

Now let us study lifts $g : B \to A$ of $\bar{g}$ to $\mathfrak{A}$, and show that there is a choice of lift g such that both fg and gf are idempotent. *A priori*, for an arbitrary fixed lift g_0, the difference

$$fg_0 - fg_0fg_0$$

is in $\mathrm{rad}_{\mathfrak{A}}(B,B)$. By axiom 1.2 (1), it is therefore nilpotent. By [Ki, proof of Cor. 7.8], there is a polynomial $P \in \mathbb{Z}[X]$ satisfying $P(1) = 1$, and such that the composition

$$fg_0P(fg_0) : B \longrightarrow B$$

is idempotent. Put

$$g_1 := g_0P(fg_0) : B \longrightarrow A \;.$$

Since $P(1) = 1$ and $\bar{f}\bar{g}$ is idempotent, the polynomial expression $P(fg_0)$ lifts $r \cdot \mathrm{id}_B + (1-r)\bar{f}\bar{g}$, for some integer r. Since $\bar{g}\bar{f}\bar{g} = \bar{g}$, the morphism g_1 therefore lifts $\bar{g}$. Furthermore,

$$fg_1 = fg_0P(fg_0)$$

is idempotent. Put

$$g := g_1fg_1 \;.$$

Then g continues to lift $\bar{g}$. Let us show that both fg and gf are idempotent:

$$(fg)^2 = (fg_1)^4 = (fg_1)^2 = fg \;,$$

$$(gf)^2 = (g_1f)^4 = g_1(fg_1)^3f = g_1(fg_1)f = gf \;.$$

To finish the proof, put

$$A^r := \ker(gf) \quad \text{and} \quad A^s := \mathrm{im}(gf) \;,$$

$$B^r := \ker(fg) \quad \text{and} \quad B^s := \mathrm{im}(fg) \;,$$

for a choice of g such that fg and gf are idempotent. The morphism f obviously repects these decompositions: $f = f^r \oplus f^s$. Modulo $\mathrm{rad}_{\mathfrak{A}}$, they yield the decompositions

$$A = \ker(\bar{f}) \oplus \bar{A}^s \;,\; B = \bar{B}^r \oplus \mathrm{im}(\bar{f})$$

fixed further above. On these, f^r induces the zero morphism, hence f^r is in the radical, while f^s induces an isomorphism, hence it is an isomorphism (recall that the projection $\mathfrak{A} \longrightarrow\!\!\!\!\rightarrow \overline{\mathfrak{A}}$ is conservative).

This proves properties (a)–(c) for the decompositions we constructed. Conversely, any pair of decompositions satisfying (a)–(c) induces decompositions

$$A = \ker(\bar{f}) \oplus \bar{A}^s \ , \ B = \bar{B}^r \oplus \operatorname{im}(\bar{f}) \quad \text{in} \quad \overline{\mathfrak{A}} \ .$$

Thus, the isomorphism classes of the components of the decompositions are unique in $\overline{\mathfrak{A}}$. But then the same is true for their isomorphism classes in $\mathfrak{A}$, thanks to fullness and conservativity of the projection $\mathfrak{A} \longrightarrow\!\!\!\!\rightarrow \overline{\mathfrak{A}}$. **q.e.d.**

Remark 1.4. (a) The decompositions

$$A = A^r \oplus A^s \ , \ B = B^r \oplus B^s$$

themselves, *i.e.*, the associated idempotent endomorphisms of A and of B, should in general not be expected to be unique.
(b) The use of [Ki, proof of Cor. 7.8] in the proof of Theorem 1.3 should be seen as a quantitative version of [R, 1.1.28]: let R be a ring, and $\mathfrak{n} \subset R$ a nil-ideal. Then every idempotent element of the quotient $R/\mathfrak{n}$ admits an idempotent lift to R.

Let us now fix an F-linear full, dense sub-category $\mathfrak{B}$ of $\mathfrak{A}$. Denote by

$$i_* : \mathfrak{B} \hookrightarrow \mathfrak{A}$$

the fully faithful inclusion. Recall that by definition, both $\mathfrak{A}$ and $\mathfrak{B}$ being F-linear, they admit finite direct sums and products, and both notions coincide [AK, Sect. 1.1]; their formation commutes therefore with i_*. Denote by

$$j^* : \mathfrak{A} \longrightarrow\!\!\!\!\rightarrow \mathfrak{A}/\mathfrak{B}$$

the projection of $\mathfrak{A}$ onto the quotient category $\mathfrak{A}/\mathfrak{B}$. Thus, j^* is full and essentially surjective. Two morphisms f_1, f_2 in $\mathfrak{A}$ yield identical images j^*f_1, j^*f_2 in $\mathfrak{A}/\mathfrak{B}$ if and only if their difference $f_2 - f_1$ factors through i_*B, for an object B of $\mathfrak{B}$.

Corollary 1.5. *In the above situation, assume $\mathfrak{A}$ to be pseudo-Abelian, and $\mathfrak{B}$ to be semi-primary. Let A be an object of $\mathfrak{A}$ admitting no non-zero direct factor belonging to $\mathfrak{B}$.*
(a) For any object B of $\mathfrak{B}$,

$$\operatorname{Hom}_{\mathfrak{A}}(A, i_*B) = \operatorname{rad}_{\mathfrak{A}}(A, i_*B) \quad \textit{and} \quad \operatorname{Hom}_{\mathfrak{A}}(i_*B, A) = \operatorname{rad}_{\mathfrak{A}}(i_*B, A) \ .$$

(b) Let A' be a second object of $\mathfrak{A}$ admitting no non-zero direct factor belonging to $\mathfrak{B}$. Then the surjection

$$j^* : \operatorname{Hom}_{\mathfrak{A}}(A', A) \longrightarrow\!\!\!\!\rightarrow \operatorname{Hom}_{\mathfrak{A}/\mathfrak{B}}(j^*A', j^*A)$$

detects isomorphisms. It also detects elements of the radical; more precisely,

$$(j^*)^{-1}\big(\mathrm{rad}_{\mathfrak{A}/\mathfrak{B}}(j^*A', j^*A)\big) = \mathrm{rad}_{\mathfrak{A}}(A', A) \ .$$

(c) Any object A' of $\mathfrak{A}$ satisfying $j^*A' \cong j^*A$ is isomorphic to a direct sum $A \oplus i_*B$, for some object B of $\mathfrak{B}$. More precisely, given an isomorphism

$$f : j^*A' \xrightarrow{\sim} j^*A \ ,$$

the isomorphism

$$A' \xrightarrow{\sim} A \oplus i_*B$$

can be chosen such that the associated projection onto the first factor

$$A' \twoheadrightarrow A$$

is a pre-image of f under j^*.

(d) Let A' be an object of $\mathfrak{A}$ satisfying $j^*A' \cong j^*A$, and admitting no non-zero direct factor belonging to $\mathfrak{B}$. Then $A' \cong A$. More precisely, any pre-image under j^* of an isomorphism $j^*A' \xrightarrow{\sim} j^*A$ is an isomorphism $A' \xrightarrow{\sim} A$.

Proof. As for (a), let $f : A \to i_*B$ and $g : i_*B \to A$ be arbitrary morphisms between A and i_*B, for an object B of $\mathfrak{B}$. It suffices to show that the composition fg belongs to the radical of i_*B. The category $\mathfrak{B}$ is dense in $\mathfrak{A}$, hence it is pseudo-Abelian. Thus, Theorem 1.3 can be applied. It yields decompositions

$$B_1^r \oplus B_1^s = B = B_2^r \oplus B_2^s \ ,$$

which are respected by fg, such that

$$(fg)^r : B_1^r \longrightarrow B_2^r$$

belongs to the radical, and

$$(fg)^s : B_1^s \xrightarrow{\sim} B_2^s \ .$$

In particular, $i_*B_1^s$ becomes a direct factor of A *via*

$$i_*B_1^s \hookrightarrow i_*B \xrightarrow{g} A$$

and

$$A \xrightarrow{f} i_*B \twoheadrightarrow i_*B_2^s \xrightarrow{((fg)^s)^{-1}} i_*B_1^s \ .$$

Our assumption on the object A implies that $B_1^s = 0 = B_2^s$, hence

$$fg = (fg)^r \in \mathrm{rad}_{\mathfrak{B}}(B, B) = \mathrm{rad}_{\mathfrak{A}}(i_*B, i_*B) \ .$$

Now let us turn to part (b) of the claim. Let $f : A' \to A$ be a morphism, such that j^*f is an isomorphism. The functor j^* being full, we are reduced to assuming that $A' = A$, and that $j^*f = \mathrm{id}_{j^*A}$. Hence the difference $\mathrm{id}_A - f$ factors through an object of $\mathfrak{B}$. By part (a), this difference belongs to the radical. By definition of the latter, f is an automorphism. Next, assume that $g : A' \to A$ is a morphism, such that j^*g belongs to the radical. For any

$h : A \to A'$, the difference $\mathrm{id}_{A'} - hg$ thus maps to an automorphism under j^*. By what we have just proved, it it therefore itself an automorphism. This shows that g belongs to the radical. We have shown that

$$(j^*)^{-1}\big(\mathrm{rad}_{\mathfrak{A}/\mathfrak{B}}(j^*A', j^*A)\big) \subset \mathrm{rad}_{\mathfrak{A}}(A', A) \; .$$

In fact, this inclusion is an equality: j^* is full, and therefore (*cf.* [AK, Lemme 1.4.7]) it maps the radical of $\mathfrak{A}$ to the radical of $\mathfrak{A}/\mathfrak{B}$.

To prove (c), let A' be in $\mathfrak{A}$, and assume $j^*A' \cong j^*A$. The functor j^* being full, there exist

$$f : A \longrightarrow A' \quad \text{and} \quad g : A' \longrightarrow A$$

such that $j^*(gf) = \mathrm{id}_{j^*A}$ and $j^*(fg) = \mathrm{id}_{j^*A'}$. From (b), we conclude that gf is an isomorphism, *i.e.*, A is indeed a direct factor of A':

$$A' \stackrel{\sim}{\longrightarrow} A \oplus B \; ,$$

and this isomorphism maps to the identity under j^*. It follows that the identity on B maps to zero, meaning (since $\mathfrak{B}$ is dense in $\mathfrak{A}$) that B is an object of $\mathfrak{B}$.

Part (d) of the claim is a formal consequence of any of (b) or (c). **q.e.d.**

Corollary 1.5 (a) motivates the following.

Definition 1.6. (a) Denote by $\mathfrak{g}$ the two-sided ideal of $\mathfrak{A}$ generated by

$$\mathrm{Hom}_{\mathfrak{A}}(A, i_*B) \quad \text{and} \quad \mathrm{Hom}_{\mathfrak{A}}(i_*B, A) \; ,$$

for all objects (A, B) of $\mathfrak{A} \times \mathfrak{B}$, such that A admits no non-zero direct factor belonging to $\mathfrak{B}$. Denote by $\mathfrak{g}_{\mathfrak{B}}$ the restriction of $\mathfrak{g}$ to $\mathfrak{B}$.
(b) Denote by $\mathfrak{A}^u$ the quotient category $\mathfrak{A}/\mathfrak{g}$, and by $\mathfrak{B}^u$ the quotient category $\mathfrak{B}/\mathfrak{g}_{\mathfrak{B}}$.

By definition, the inclusion $i_* : \mathfrak{B} \hookrightarrow \mathfrak{A}$ induces a fully faithful functor $\mathfrak{B}^u \hookrightarrow \mathfrak{A}^u$. Slightly abusing notation, we shall denote this functor by the same symbol i_*. The ideal $\mathfrak{g}$ should be thought of as containing "gluing data" between $\mathfrak{B}$ and $\mathfrak{A}/\mathfrak{B}$. The categories $\mathfrak{A}^u$ and $\mathfrak{B}^u$ are the "un-glued" quotients of $\mathfrak{A}$ and $\mathfrak{B}$, respectively. A morphism is in $\mathfrak{g}$ if and only if it is the sum of morphisms g_n, each of which factors through an element of $\mathrm{Hom}_{\mathfrak{A}}(A_n, i_*B_n)$ or of $\mathrm{Hom}_{\mathfrak{A}}(i_*B_n, A_n)$, for A_n and B_n as in Definition 1.6 (a). In particular, $\mathfrak{g}$ is contained in the kernel of j^*. Under the hypotheses of Corollary 1.5, it is also contained in $\mathrm{rad}_{\mathfrak{A}}$ (Corollary 1.5 (a)).

Corollary 1.7. *Keep the situation of Corollary 1.5,* i.e., *assume $\mathfrak{A}$ to be pseudo-Abelian, $\mathfrak{B}$ to be semi-primary, and A to be an object of $\mathfrak{A}$ admitting no non-zero direct factor belonging to $\mathfrak{B}$. In addition, assume that there exists an object B_0 of $\mathfrak{B}$, such that any endomorphism f of A with $j^*f = 0$ factors through i_*B_0.*
(a) The kernel of

$$j^* : \mathrm{End}_{\mathfrak{A}}(A) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{End}_{\mathfrak{A}/\mathfrak{B}}(j^*A)$$

equals $\mathfrak{g}(A,A)$*. It is is a nilpotent ideal contained in* $\mathrm{rad}_{\mathfrak{A}}(A,A)$*. In particular, any idempotent of* $\mathrm{End}_{\mathfrak{A}/\mathfrak{B}}(j^*A)$ *can be lifted to an idempotent of* $\mathrm{End}_{\mathfrak{A}}(A)$*.*
(b) For any object B *of* $\mathfrak{B}$*,*

$$\mathfrak{g}(A \oplus i_*B, A \oplus i_*B)$$

is a nilpotent ideal contained in $\mathrm{rad}_{\mathfrak{A}}(A \oplus i_*B, A \oplus i_*B)$*.*

Proof. Let f be an endomorphism of A such that $j^*f = 0$. By our assumption, f factors through i_*B_0: there are morphisms $g : A \to i_*B_0$ and $h : i_*B_0 \to A$ such that $f = hg$. Since $\mathfrak{B}$ is semi-primary, there is a positive integer N such that $\mathrm{rad}_{\mathfrak{A}}(i_*B_0, i_*B_0)^N = \mathrm{rad}_{\mathfrak{B}}(B_0, B_0)^N = 0$. Let $f_1, \ldots, f_{N+1}$ be endomorphisms of A with trivial j^*f_k, and choose factorizations

$$f_k : A \xrightarrow{g_k} i_*B_0 \xrightarrow{h_k} A$$

as above, $k = 1, \ldots, N+1$. Then

$$(g_1h_2)(g_2h_3)\cdots(g_Nh_{N+1}) = 0 : i_*B_0 \longrightarrow i_*B_0 \ .$$

Therefore, $f_1f_2\cdots f_{N+1} = 0 : A \to A$. This proves part (a) of our claim.

As for part (b), in order to show that

$$\mathfrak{g}(A \oplus i_*B, A \oplus i_*B)$$

is nilpotent, it suffices to show that the analogous statement holds for both $\mathfrak{g}(A,A)$ and $\mathfrak{g}(i_*B, i_*B)$ [AK, proof of Prop. 2.3.4 c)]. Nilpotency of $\mathfrak{g}(A,A)$ was proved in (a). For

$$\mathfrak{g}(i_*B, i_*B) = \mathfrak{g}_{\mathfrak{B}}(B,B) \subset \mathrm{rad}_{\mathfrak{B}}(B,B) \ ,$$

use semi-primality of $\mathfrak{B}$. **q.e.d.**

The hypotheses from Corollary 1.7 naturally lead to the following.

Definition 1.8. Let us denote the following hypothesis by $(*)$:
Every object of $\mathfrak{A}/\mathfrak{B}$ admits a pre-image A under j^* without non-zero direct factors belonging to $\mathfrak{B}$, and an object B_0 of $\mathfrak{B}$, such that any endomorphism f of A with $j^*f = 0$ factors through i_*B_0.

As we shall in Section 2, hypothesis $(*)$ is satisfied in the context of weight structures. Putting everything together, we get the following results.

Theorem 1.9. *Let* $i_* : \mathfrak{B} \hookrightarrow \mathfrak{A}$ *be the inclusion of one* F*-linear pseudo-Abelian category into another, with* $\mathfrak{B}$ *full and dense in* $\mathfrak{A}$*. Denote by* $j^* : \mathfrak{A} \twoheadrightarrow \mathfrak{A}/\mathfrak{B}$ *the projection onto the quotient category. Assume hypothesis 1.8* $(*)$*, and that* $\mathfrak{B}$ *is semi-primary.*
(a) Any object of $\mathfrak{A}$ *is isomorphic to a direct sum* $A \oplus i_*B$*, for some object* (A,B) *of* $\mathfrak{A} \times \mathfrak{B}$*, such that* A *admits no non-zero direct factor belonging to* $\mathfrak{B}$*. The object* (A,B) *is unique up to an isomorphism, which becomes unique*

in $\mathfrak{A}^u \times \mathfrak{B}^u$ *(and hence in* $\mathfrak{A}/\mathfrak{B} \times \mathfrak{B}^u$*).*

(b) $\mathfrak{g}$ *is a nilpotent ideal contained in* $\mathrm{rad}_{\mathfrak{A}} \cap \ker j^*$*.*

(c) The F*-linear categories* $\mathfrak{A}/\mathfrak{B}$*,* $\mathfrak{B}^u$ *and* $\mathfrak{A}^u$ *are pseudo-Abelian.*

(d) The functors j^* *and* i_* *induce a canonical equivalence of* F*-linear categories*

$$\mathfrak{A}^u \stackrel{\sim}{\longrightarrow} \mathfrak{A}/\mathfrak{B} \times \mathfrak{B}^u \ .$$

(e) The equivalence from part (d) descends further to induce a canonical equivalence of F*-linear categories*

$$\overline{\mathfrak{A}} \stackrel{\sim}{\longrightarrow} \overline{\mathfrak{A}/\mathfrak{B}} \times \overline{\mathfrak{B}} \ .$$

(f) If $\mathfrak{B}$ *and* $\mathfrak{A}/\mathfrak{B}$ *are semi-primary, then so is* $\mathfrak{A}$*.*

Proof. The first assertion of part (a) follows from hypothesis 1.8 $(*)$ and Corollary 1.5 (c). The second assertion of part (a) is proved together with part (d): recall that by definition of $\mathfrak{g}$,

$$\mathrm{Hom}_{\mathfrak{A}^u}(A_1 \oplus i_*B_1, A_2 \oplus i_*B_2) = \mathrm{Hom}_{\mathfrak{A}^u}(A_1, A_2) \oplus \mathrm{Hom}_{\mathfrak{A}^u}(i_*B_1, i_*B_2)$$

(for (A_1, B_1) and (A_2, B_2) in $\mathfrak{A} \times \mathfrak{B}$, such that the A_k do not admit non-zero direct factors belonging to $\mathfrak{B}$). Again by definition, the right hand side equals

$$\mathrm{Hom}_{\mathfrak{A}/\mathfrak{B}}(j^*A_1, j^*A_2) \oplus \mathrm{Hom}_{\mathfrak{B}^u}(B_1, B_2) \ .$$

Part (b) follows from (a), and from Corollary 1.7 (b).

The category $\mathfrak{A}/\mathfrak{B}$ is pseudo-Abelian thanks to Corollary 1.7 (a) (take $B = 0$ in the decomposition from (a)). The categories $\mathfrak{B}^u$ and $\mathfrak{A}^u$ are pseudo-Abelian since by (b), $\mathfrak{g}$ is nilpotent. This proves part (c) of the claim.

As for parts (e) and (f), note that according to (b), the ideal $\mathfrak{g}$ is contained in $\mathrm{rad}_{\mathfrak{A}}$. It follows that $\mathfrak{g}_{\mathfrak{B}}$ is contained in $\mathrm{rad}_{\mathfrak{B}}$. Therefore, the category $\mathfrak{B}^u$ is semi-primary, and $i_* : \mathfrak{B}^u \hookrightarrow \mathfrak{A}^u$ satisfies the same hypotheses as $i_* : \mathfrak{B} \hookrightarrow \mathfrak{A}$. Parts (b) and (d) then allow us to assume that

$$\mathfrak{A} = \mathfrak{A}/\mathfrak{B} \times \mathfrak{B} \ ,$$

in which case claims (e) and (f) are trivial. **q.e.d.**

Remark 1.10. Assume that F is a finite direct product of perfect fields. According to [AK, Déf. 2.4.1], the F-linear category $\mathfrak{A}$ is a *Wedderburn category* if it is semi-primary, and if for all objects A of $\mathfrak{A}$, the F-algebra $\mathrm{End}_{\overline{\mathfrak{A}}}(A)$ is a finite F-module. In particular, every object of $\overline{\mathfrak{A}}$ is then of finite length. Obviously, Theorem 1.3, Corollaries 1.5 and 1.7, and Theorems 1.9 (a)–(e) continue to hold when the respective hypotheses on semi-primality are replaced by the (more restrictive) hypotheses of being Wedderburn. It follows from the equivalence

$$\overline{\mathfrak{A}} \stackrel{\sim}{\longrightarrow} \overline{\mathfrak{A}/\mathfrak{B}} \times \overline{\mathfrak{B}}$$

that Theorem 1.9 (f) admits such a variant, too. More precisely, keeping all the hypotheses of Theorem 1.9 as they are, but supposing $\mathfrak{B}$ and $\mathfrak{A}/\mathfrak{B}$ to be Wedderburn categories, then $\mathfrak{A}$ is a Wedderburn category.

2 The abstract theory of intermediate extensions

We keep the coefficients F fixed in the beginning, and consider an F-linear pseudo-Abelian triangulated category $\mathcal{C}$. Recall the following definitions.

Definition 2.1 ([B2, Def. 1.3.1]). (a) A *weight structure on* $\mathcal{C}$ is a pair $w = (\mathcal{C}_{w\le 0}, \mathcal{C}_{w\ge 0})$ of full sub-categories of $\mathcal{C}$, such that, putting

$$\mathcal{C}_{w\le n} := \mathcal{C}_{w\le 0}[n] \quad , \quad \mathcal{C}_{w\ge n} := \mathcal{C}_{w\ge 0}[n] \quad \forall\, n \in \mathbb{Z} \, ,$$

the following conditions are satisfied.

(1) The categories $\mathcal{C}_{w\le 0}$ and $\mathcal{C}_{w\ge 0}$ are dense in $\mathcal{C}$.

(2) (Semi-invariance with respect to shifts.) We have the inclusions

$$\mathcal{C}_{w\le 0} \subset \mathcal{C}_{w\le 1} \quad , \quad \mathcal{C}_{w\ge 0} \supset \mathcal{C}_{w\ge 1}$$

of full sub-categories of $\mathcal{C}$.

(3) (Orthogonality.) For any pair of objects $X \in \mathcal{C}_{w\le 0}$ and $Y \in \mathcal{C}_{w\ge 1}$, we have

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) = 0 \, .$$

(4) (Weight filtration.) For any object $M \in \mathcal{C}$, there exists an exact triangle

$$X \longrightarrow M \longrightarrow Y \longrightarrow X[1]$$

in $\mathcal{C}$, such that $X \in \mathcal{C}_{w\le 0}$ and $Y \in \mathcal{C}_{w\ge 1}$.

(b) Let w be a weight structure on $\mathcal{C}$. The *heart of* w is the full F-linear sub-category $\mathcal{C}_{w=0}$ of $\mathcal{C}$ whose objects lie both in $\mathcal{C}_{w\le 0}$ and in $\mathcal{C}_{w\ge 0}$.

Slightly generalizing the terminology, for $n \in \mathbb{Z}$, we shall refer to any exact triangle

$$X \longrightarrow M \longrightarrow Y \longrightarrow X[1]$$

in $\mathcal{C}$, with $X \in \mathcal{C}_{w\le n}$ and $Y \in \mathcal{C}_{w\ge n+1}$, as a weight filtration of M.

We shall be particularly interested in weight structures whose hearts are semi-primary. In this context, Theorem 1.3 will be used as follows.

Theorem 2.2. *Let w be a weight structure on $\mathcal{C}$, and assume $\mathcal{C}_{w=0}$ to be semi-primary.*
(a) Any object M of $\mathcal{C}$ admits a minimal weight filtration, i.e., *a weight filtration*

$$M_{\le 0} \longrightarrow M \longrightarrow M_{\ge 1} \xrightarrow{\delta} M_{\le 0}[1]$$

($M_{\le 0} \in \mathcal{C}_{w\le 0}$, $M_{\ge 1} \in \mathcal{C}_{w\ge 1}$) satisfying

$$\delta \in \mathrm{rad}_{\mathcal{C}}(M_{\ge 1}, M_{\le 0}[1]) \ .$$

(b) Any two minimal weight filtrations of the same object M of $\mathcal{C}$ are isomorphic, the isomorphism being unique up to adding morphisms in $\mathrm{rad}_{\mathcal{C}}$. *The minimal weight filtration of M is a direct factor of any other weight filtration of M.*
(c) Minimal weight filtrations are "functorial up to $\mathrm{rad}_{\mathcal{C}}$*": let*

$$M_{\le 0} \longrightarrow M \longrightarrow M_{\ge 1} \xrightarrow{\delta_M} M_{\le 0}[1]$$

and

$$N_{\le 0} \longrightarrow N \longrightarrow N_{\ge 1} \xrightarrow{\delta_N} N_{\le 0}[1]$$

be minimal weight filtrations of objects M and N of $\mathcal{C}$, and $\alpha : M \to N$ a morphism. Then α extends to give a morphism of exact triangles

$$\begin{tikzcd} M_{\le 0} \arrow[r] \arrow[d, "\alpha_{\le 0}"'] & M \arrow[r] \arrow[d, "\alpha"'] & M_{\ge 1} \arrow[r, "\delta_M"] \arrow[d, "\alpha_{\ge 1}"'] & M_{\le 0}[1] \arrow[d, "\alpha_{\le 0}[1]"] \\ N_{\le 0} \arrow[r] & N \arrow[r] & N_{\ge 1} \arrow[r, "\delta_N"] & N_{\le 0}[1] \end{tikzcd}$$

and $\alpha_{\le 0}$ and $\alpha_{\ge 1}$ are unique up to adding morphisms in $\mathrm{rad}_{\mathcal{C}}$.

Proof. For part (a) of the claim, start by choosing *any* weight filtration

$$X \longrightarrow M \longrightarrow Y \longrightarrow X[1]$$

of M, with $X \in \mathcal{C}_{w\le -1}$ and $Y \in \mathcal{C}_{w\ge 0}$. We leave it to the reader to construct from a minimal weight filtration for Y a minimal weight filtration for M. Thus we may assume that $M \in \mathcal{C}_{w\ge 0}$. Similarly, starting from a weight filtration

$$X \longrightarrow M \longrightarrow Y \longrightarrow X[1]$$

of M, with $X \in \mathcal{C}_{w\ge 0} \cap \mathcal{C}_{w\le 1}$ and $Y \in \mathcal{C}_{w\ge 2}$, we may assume that $M \in \mathcal{C}_{w\ge 0} \cap \mathcal{C}_{w\le 1}$. This means that M is actually a cone of a morphism in $\mathcal{C}_{w=0}$:

$$A \xrightarrow{f} B \longrightarrow M \longrightarrow A[1]$$

(A and B in $\mathcal{C}_{w=0}$). Now apply Theorem 1.3 to see that M is also the cone of a morphism f^r in the radical.

Claim (b) is a special case of claim (c). Using orthogonality (*c.f.* [B1, Lemma 1.4.1 1]), it is easy to see that any morphism $\alpha : M \to N$ can be

extended to a morphism of minimal (in fact, of arbitrary) weight filtrations. The difference of two choices of extensions $(\alpha^k_{\le 0}, \alpha^k_{\ge 1})$, $k = 1, 2$, makes the following diagram commute.

$$\begin{tikzcd} M_{\le 0} \arrow[r] \arrow[d, "\alpha^2_{\le 0}-\alpha^1_{\le 0}"'] & M \arrow[r] \arrow[d, "0"'] & M_{\ge 1} \arrow[r, "\delta_M"] \arrow[d, "\alpha^2_{\ge 1}-\alpha^1_{\ge 1}"'] & M_{\le 0}[1] \arrow[d, "\alpha^2_{\le 0}[1]-\alpha^1_{\le 0}[1]"] \\ N_{\le 0} \arrow[r] & N \arrow[r] & N_{\ge 1} \arrow[r, "\delta_N"] & N_{\le 0}[1] \end{tikzcd}$$

Recall that both δ_M and δ_N are in the radical, Thus, both differences $\alpha^2_{\le 0} - \alpha^1_{\le 0}$ and $\alpha^2_{\ge 1} - \alpha^1_{\ge 1}$ factor through a morphism in $\mathrm{rad}_{\mathcal{C}}$. The radical being an ideal, they therefore belong to $\mathrm{rad}_{\mathcal{C}}$, too. **q.e.d.**

Examples 2.3. (a) Let w be a weight structure on $\mathcal{C}$, and assume in addition that $\mathcal{C}$ carries a bounded t-structure $t = (\mathcal{C}^{t\le 0}, \mathcal{C}^{t\ge 0})$, which is *transversal* to w [B3, Def. 1.2.2]. According to [B3, Thm. 1.2.1 (i), Rem. 1.2.3 2.], this implies in particular that the full sub-categories given by the intersections

$$\mathcal{C}^{t=n}_{w=0} := \mathcal{C}_{w=0} \cap \mathcal{C}^{t=n}$$

are Abelian semi-simple, for all $n \in \mathbb{Z}$, and that any object of $\mathcal{C}_{w=0}$ is isomorphic to a finite direct sum of objects in $\mathcal{C}^{t=n}_{w=0}$, for varying n. [AK, proof of Prop. 2.3.4 c)] implies (*c.f.* the proof of Corollary 1.7 (b)) that the heart $\mathcal{C}_{w=0}$ is semi-primary.

(b) Let X be a scheme over $\mathbb{C}$, assume F to be a field contained in $\mathbb{R}$, and put $\mathcal{C} := D^b(\mathbf{MHM}_F\, X)$, the bounded derived category of *algebraic mixed F-Hodge modules* on X [Sa2, Sect. 4.2]. According to [B3, Prop. 2.3.1 I], $\mathcal{C}$ carries a canonical weight structure w: indeed, the categories $D^b(\mathbf{MHM}_F\, X)_{w\le 0}$ and $D^b(\mathbf{MHM}_F\, X)_{w\ge 0}$ are equal to the sub-categories of complexes of Hodge modules of weights ≤ 0 and ≥ 0 (in the sense of [Sa2, Def. 4.5]), respectively. Furthermore, the canonical t-structure on $D^b(\mathbf{MHM}_F\, X)$ is transversal to w. From (a), we conclude that $D^b(\mathbf{MHM}_F\, X)_{w=0}$ is semi-primary. Since $D^b(\mathbf{MHM}_F\, X)$ is pseudo-Abelian, Theorem 2.2 can be applied, *i.e.*, minimal weight filtrations exist. This seems to have non-trivial consequences even for $X = \mathbf{Spec}\, \mathbb{C}$ (in which case $\mathbf{MHM}_F\, X = \mathbf{MHS}_F$, the category of mixed graded-polarizable F-Hodge structures). We intend to exploit these elsewhere.

(c) Minimal weight filtrations do not exist for arbitrary weight structures. Take an Abelian F-linear category $\mathfrak{A}$, and consider the bounded homotopy category $\mathcal{C} = K^b(\mathfrak{A})$ with the "stupid" weight structure

$$w = \big(K^{b,\ge 0}(\mathfrak{A}), K^{b,\le 0}(\mathfrak{A})\big)$$

[B1, p. 400]. We leave it to the reader to verify that the existence of minimal weight filtrations in $\mathcal{C}$ implies (and is in fact equivalent to) the validity of the conclusion of Theorem 1.3 for any morphism f in $\mathfrak{A}$. Take $\mathfrak{A}$:= the category of $F[X]$-modules and f := multiplication by X on $F[X]$ to see that this conclusion is not valid for general Abelian F-linear categories. (According to

[AK, Prop. 2.3.5], it *is* valid for Abelian F-linear categories in which every object is of finite length.)

Remark 2.4. In our previous work [W2], we followed a different approach to minimal weight filtrations. Instead of deducing their existence from abstract properties of the category $\mathcal{C}$, we basically restricted our attention to objects M admitting a minimal weight filtration for an obvious reason. Indeed, for integers $m \le n$, M is said to admit a weight filtration *avoiding weights* $m, \ldots, n$ [W2, Def. 1.6] if there exists an exact triangle

$$M_{\le m-1} \longrightarrow M \longrightarrow M_{\ge n+1} \stackrel{\delta}{\longrightarrow} M_{\le m-1}[1]$$

in $\mathcal{C}$, with $M_{\le m-1} \in \mathcal{C}_{w \le m-1}$ and $M_{\ge n+1} \in \mathcal{C}_{w \ge n+1}$. The integer $m = (m-1)+1$ being strictly smaller than $n+1$, orthogonality (condition (3) in Definition 2.1 (a)) tells us that there are no non-zero morphisms from $M_{\le m-1}[1]$ to $M_{\ge n+1}$. Therefore, the morphism δ is necessarily in the radical of $\mathcal{C}$. It follows that for objects M of $\mathcal{C}$ admitting a weight filtration avoiding weight 0 or weight 1, that weight filtration is minimal in the sense of Theorem 2.2 (a). Note that in this case, there are proper unicity and functoriality statements concerning the minimal weight filtrations [W2, Prop. 1.7], *i.e.*, the passage to the quotient by $\mathrm{rad}_{\mathcal{C}}$ as in Theorem 2.2 (b) and (c) is not needed. The draw-back of the method from [W2] is that in the applications to motives (see *e.g.* [W3]), the condition on absence of weights is not necessarily easy to verify.

For the rest of this section, let us assume that our F-linear triangulated category is obtained by *gluing*; modifying slightly our notation, let us fix three F-linear pseudo-Abelian triangulated categories $\mathcal{C}(U)$, $\mathcal{C}(X)$ and $\mathcal{C}(Z)$, together with six exact functors

$$\mathcal{C}(U) \xrightarrow{j_!} \mathcal{C}(X) \xrightarrow{i^*} \mathcal{C}(Z)$$

$$\mathcal{C}(U) \xleftarrow{j^*} \mathcal{C}(X) \xleftarrow{i_*} \mathcal{C}(Z)$$

$$\mathcal{C}(U) \xrightarrow{j_*} \mathcal{C}(X) \xrightarrow{i^!} \mathcal{C}(Z)$$

satisfying the axioms from [BBD, Sect. 1.4.3]. We assume that $\mathcal{C}(U)$, $\mathcal{C}(X)$ and $\mathcal{C}(Z)$ are equipped with weight structures w (the same letter for the three weight structures), and that the one on $\mathcal{C}(X)$ is actually obtained from the two others in a way *compatible with the gluing*, meaning that the left adjoints $j_!$, j^*, i^* and i_* respect the categories $\mathcal{C}(\bullet)_{w\le 0}$, and the right adjoints j^*, j_*, i_* and $i^!$ respect the categories $\mathcal{C}(\bullet)_{w\ge 0}$. In particular, we have a fully faithful functor

$$i_* : \mathcal{C}(Z)_{w=0} \longrightarrow \mathcal{C}(X)_{w=0}$$

compatible with the formation of finite direct sums, and with dense image. We also have an exact functor

$$j^* : \mathcal{C}(X)_{w=0} \longrightarrow \mathcal{C}(U)_{w=0} \ .$$

Proposition 2.5. *The functor*

$$j^* : \mathcal{C}(X)_{w=0} \longrightarrow \mathcal{C}(U)_{w=0}$$

is full and essentially surjective.

Proof. We first establish essential surjectivity. Let M_U be an object of $\mathcal{C}(U)_{w=0}$, and consider the morphism

$$m : j_! M_U \longrightarrow j_* M_U$$

in $\mathcal{C}(X)$ [BBD, Axiom (1.4.3.2)]. Applying j^* to m yields an isomorphism [BBD, Axiom (1.4.3.5)]. Therefore, by localization [BBD, Axiom (1.4.3.4)], any cone of m is of the form $i_* C$, for an object C of $\mathcal{C}(Z)$. Choose and fix such a cone $i_* C$, as well as a weight filtration

$$C_{\le 0} \xrightarrow{c_-} C \xrightarrow{c_+} C_{\ge 1} \xrightarrow{\delta} C_{\le 0}[1]$$

of $C \in \mathcal{C}(Z)$. Thus,

$$C_{\le 0} \in \mathcal{C}(Z)_{w \le 0} \quad \text{and} \quad C_{\ge 1} \in \mathcal{C}(Z)_{w \ge 1} \ .$$

According to axiom TR4' of triangulated categories (see [BBD, Sect. 1.1.6] for an equivalent formulation), the diagram of exact triangles

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & i_* C_{\ge 1}[-1] \arrow[r, equal] & i_* C_{\ge 1}[-1] \arrow[r] \arrow[d, "i_*\delta[-1]"] & 0 \arrow[d] \\
j_! M_U \arrow[d, equal] & & i_* C_{\le 0} \arrow[r] \arrow[d, "i_* c_-"] & j_! M_U[1] \arrow[d, equal] \\
j_! M_U \arrow[r, "m"] \arrow[d] & j_* M_U \arrow[r] \arrow[d] & i_* C \arrow[r] \arrow[d, "i_* c_+"] & j_! M_U[1] \arrow[d] \\
0 \arrow[r] & i_* C_{\ge 1} \arrow[r, equal] & i_* C_{\ge 1} \arrow[r] & 0
\end{tikzcd}$$

in $\mathcal{C}(X)$ can be completed to give

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & i_* C_{\ge 1}[-1] \arrow[r, equal] \arrow[d] & i_* C_{\ge 1}[-1] \arrow[r] \arrow[d, "i_*\delta[-1]"] & 0 \arrow[d] \\
j_! M_U \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & i_* C_{\le 0} \arrow[r] \arrow[d, "i_* c_-"] & j_! M_U[1] \arrow[d, equal] \\
j_! M_U \arrow[r, "m"] \arrow[d] & j_* M_U \arrow[r] \arrow[d] & i_* C \arrow[r] \arrow[d, "i_* c_+"] & j_! M_U[1] \arrow[d] \\
0 \arrow[r] & i_* C_{\ge 1} \arrow[r, equal] & i_* C_{\ge 1} \arrow[r] & 0
\end{tikzcd}$$

with $M \in \mathcal{C}(X)$. Since the composition of functors $j^* i_*$ is trivial [BBD, Axiom (1.4.3.3)], the inverse image $j^* M$ is isomorphic to M_U. Now recall

that the weight structures are supposed to be compatible with the gluing: the functors $i_! = i_*$ and $j_!$ respect $\mathcal{C}(\bullet)_{w\le 0}$, and i_* and j_* respect $\mathcal{C}(\bullet)_{w\ge 0}$. Therefore, by the above diagram, the object M is simultaneously an extension of objects of weights ≤ 0, and an extension of objects of weights ≥ 0. It follows easily (*c.f.* [B1, Prop. 1.3.3 3]) that M belongs to $\mathcal{C}(X)_{w=0}$.

In order to prove that j^* is full, let M and N be objects of $\mathcal{C}(X)_{w=0}$, and assume that a morphism

$$\beta_U : j^*M \longrightarrow j^*N$$

between their restrictions to U is given. Consider the localization triangles for M and for N.

$$\begin{tikzcd} i_*i^*M[-1] \arrow[r] & j_!j^*M \arrow[r] \arrow[d, "j_!\beta_U"] & M \arrow[r] & i_*i^*M \\ i_*i^*N[-1] \arrow[r] & j_!j^*N \arrow[r] & N \arrow[r] & i_*i^*N \end{tikzcd}$$

Thanks to compatibility of the weight structures with the gluing, these triangles are weight filtrations of $j_!j^*M$ and of $j_!j^*N$, respectively. By orthogonality (condition (3) in Definition 2.1 (a)), any morphism from $i_*i^*M[-1]$ to N is zero. Therefore, the above diagram can be completed to give a morphism of exact triangles. **q.e.d.**

Corollary 2.6. *The functor*

$$j^* : \mathcal{C}(X)_{w=0} \longrightarrow\!\!\!\!\!\rightarrow \mathcal{C}(U)_{w=0}$$

identifies $\mathcal{C}(U)_{w=0}$ with the categorial quotient of $\mathcal{C}(X)_{w=0}$ by $\mathcal{C}(Z)_{w=0}$.

Proof. By [BBD, Prop. 1.4.5], the functor $j^* : \mathcal{C}(X) \longrightarrow\!\!\!\!\!\rightarrow \mathcal{C}(U)$ induces a canonical equivalence

$$\mathcal{C}(X)/\mathcal{C}(Z) \xrightarrow{\sim} \mathcal{C}(U) .$$

Since $\mathcal{C}(X)_{w=0} \cap i_*\mathcal{C}(Z) = i_*\mathcal{C}(Z)_{w=0}$, this implies that $\mathcal{C}(X)_{w=0}/\mathcal{C}(Z)_{w=0}$ becomes a full sub-category of $\mathcal{C}(U)$ *via* j^*. Now apply Proposition 2.5. **q.e.d.**

We are therefore in the context studied in Section 1, with

$$\mathfrak{B} = \mathcal{C}(Z)_{w=0} \ , \ \mathfrak{A} = \mathcal{C}(X)_{w=0} \ , \text{ and } \mathfrak{A}/\mathfrak{B} = \mathcal{C}(U)_{w=0} \ .$$

Lemma 2.7. *(a) Let $M \in \mathcal{C}(X)_{w=0}$. Assume that the composition of the two adjunction morphisms*

$$i_*i^!M \longrightarrow M \longrightarrow i_*i^*M$$

belongs to the radical

$$\mathrm{rad}_{\mathcal{C}(X)}(i_*i^!M, i_*i^*M) = \mathrm{rad}_{\mathcal{C}(Z)}(i^!M, i^*M) \ .$$

Then M admits no non-zero direct factor belonging to $i_*\mathcal{C}(Z)_{w=0}$.
(b) Let $M \in \mathcal{C}(X)_{w=0}$. Then there exists an object N_0 of $\mathcal{C}(Z)_{w=0}$, such that

any endomorphism f of M with $j^* f = 0$ factors through $i_* N_0$.
(c) Assume $\mathcal{C}(Z)_{w=0}$ to be semi-primary. Then any object M_U of $\mathcal{C}(U)_{w=0}$ admits a pre-image M under j^* such that the composition of the two adjunction morphisms

$$i_* i^! M \longrightarrow M \longrightarrow i_* i^* M$$

belongs to the radical.
(d) Assume $\mathcal{C}(Z)_{w=0}$ to be semi-primary. Then hypothesis 1.8 $(*)$ is satisfied for

$$\mathfrak{B} = \mathcal{C}(Z)_{w=0} \ , \ \mathfrak{A} = \mathcal{C}(X)_{w=0} \ , \ \textit{and}\ \mathfrak{A}/\mathfrak{B} = \mathcal{C}(U)_{w=0} \ .$$

Proof. (a): Assume that some composition of morphisms

$$i_* N \longrightarrow M \longrightarrow i_* N$$

gives the identity on $i_* N$, for some object N of $\mathcal{C}(Z)_{w=0}$. Then the adjunction properties of i^*, i_* and $i^!$ show that N is a direct factor of both $i^! M$ and $i^* M$, and that the restriction of the composition $i^! M \to i^* M$ of the adjunction morphisms to this direct factor is the identity. But by assumption, this composition belongs to $\mathrm{rad}_{\mathcal{C}(Z)}(i^! M, i^* M)$, hence $N = 0$.

(b): Choose a weight filtration

$$X \longrightarrow i^* M \longrightarrow N_0 \longrightarrow X[1]$$

in $\mathcal{C}(Z)$, with $X \in \mathcal{C}(Z)_{w \le -1}$ and $N_0 \in \mathcal{C}(Z)_{w=0}$. Any endomorphism f of M restricting trivially to $\mathcal{C}(U)_{w=0}$ factors through the adjunction morphism $M \to i_* i^* M$. Thanks to orthogonality, f actually factors through $i_* N_0$.

(c): Let M_U be an object of $\mathcal{C}(U)_{w=0}$. We repeat the proof of Proposition 2.5: consider the morphism

$$m : j_! M_U \longrightarrow j_* M_U$$

in $\mathcal{C}(X)$, and fix a cone $i_* C$ of m. Any weight filtration

$$C_{\le 0} \xrightarrow{c_-} C \xrightarrow{c_+} C_{\ge 1} \xrightarrow{\delta} C_{\le 0}[1]$$

of C yields a diagram of exact triangles

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & i_* C_{\ge 1}[-1] \arrow[r, equal] \arrow[d] & i_* C_{\ge 1}[-1] \arrow[r] \arrow[d, "i_*\delta[-1]"] & 0 \arrow[d] \\
j_! M_U \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & i_* C_{\le 0} \arrow[r] \arrow[d, "i_* c_-"] & j_! M_U[1] \arrow[d, equal] \\
j_! M_U \arrow[r, "m"] \arrow[d] & j_* M_U \arrow[r] \arrow[d] & i_* C \arrow[r] \arrow[d, "i_* c_+"] & j_! M_U[1] \arrow[d] \\
0 \arrow[r] & i_* C_{\ge 1} \arrow[r, equal] & i_* C_{\ge 1} \arrow[r] & 0
\end{tikzcd}$$

with $M \in \mathcal{C}(X)_{w=0}$. This is true in particular when the weight filtration of C is chosen as the minimal weight filtration of Theorem 2.2. In this case,

the morphism

$$\delta[-1] : C_{\geq 1}[-1] \longrightarrow C_{\leq 0}$$

belongs to the radical. But by the above diagram, this morphism is isomorphic to the composition of the two adjunction morphisms

$$i_* i^! M \longrightarrow M \longrightarrow i_* i^* M \ .$$

(d): This follows from parts (c), (a) and (b). **q.e.d.**

Remark 2.8. (a) As the proof of Proposition 2.5 shows, there is a quantitative version of essential surjectivity of

$$j^* : \mathcal{C}(X)_{w=0} \longrightarrow \mathcal{C}(U)_{w=0} \ .$$

Indeed, let M_U be an object of $\mathcal{C}(U)_{w=0}$, and

$$m : j_! M_U \longrightarrow j_* M_U$$

the morphism from [BBD, Axiom (1.4.3.2)]. Choose and fix a cone $i_* C$ of m. Then the map

$$\{(C_{\leq 0}, C_{\geq 1})\} / \cong \longrightarrow \{M\} / \cong$$

is a bijection between
(1) the isomorphism classes of weight filtrations of C,
(2) the isomorphism classes of pre-images M of M_U under j^*.
The inverse bijection maps the class of M to the class of $(i^* M, i^! M[1])$.
(b) In the situation of (a), assume $\mathcal{C}(Z)_{w=0}$ to be semi-primary. As the proof of Lemma 2.7 shows, the bijection

$$\{(C_{\leq 0}, C_{\geq 1})\} / \cong \longrightarrow \{M\} / \cong$$

from (a) maps the class of the minimal weight filtration of Theorem 2.2 to the class of an object admitting no non-zero direct factor belonging to $i_* \mathcal{C}(Z)_{w=0}$; note that according to Corollary 1.5 (d), this class is unique.

Theorem 2.9. *Let the F-linear pseudo-Abelian triangulated categories $\mathcal{C}(U)$, $\mathcal{C}(X)$ and $\mathcal{C}(Z)$ be related by gluing, and equipped with weight structures w compatible with the gluing.*
(a) If $\mathcal{C}(Z)_{w=0}$ is semi-primary, then the conclusions of Theorem 1.9 (a)–(e) hold for

$$\mathfrak{B} = \mathcal{C}(Z)_{w=0} \ , \ \mathfrak{A} = \mathcal{C}(X)_{w=0} \ , \ \textit{and} \ \mathfrak{A}/\mathfrak{B} = \mathcal{C}(U)_{w=0} \ .$$

In particular, the functors j^* and i_* then induce canonical equivalences of F-linear categories

$$\mathcal{C}(X)^u_{w=0} \xrightarrow{\sim} \mathcal{C}(U)_{w=0} \times \mathcal{C}(Z)^u_{w=0}$$

and

$$\overline{\mathcal{C}(X)_{w=0}} \xrightarrow{\sim} \overline{\mathcal{C}(U)_{w=0}} \times \overline{\mathcal{C}(Z)_{w=0}} \ .$$

(b) If both $\mathcal{C}(Z)_{w=0}$ and $\mathcal{C}(U)_{w=0}$ are semi-primary, then so is $\mathcal{C}(X)_{w=0}$. If both $\mathcal{C}(Z)_{w=0}$ and $\mathcal{C}(U)_{w=0}$ are Wedderburn categories, then so is $\mathcal{C}(X)_{w=0}$.

Proof. By Corollary 2.6 and Lemma 2.7 (d), the hypotheses of Theorem 1.9 are fulfilled. **q.e.d.**

Definition 2.10. Let the F-linear pseudo-Abelian triangulated categories $\mathcal{C}(U)$, $\mathcal{C}(X)$ and $\mathcal{C}(Z)$ be related by gluing, and equipped with weight structures w compatible with the gluing. Assume $\mathcal{C}(Z)_{w=0}$ to be semi-primary. Define the *intermediate extension*

$$j_{!*} : \mathcal{C}(U)_{w=0} \hookrightarrow \mathcal{C}(X)^u_{w=0}$$

as the fully faithful functor corresponding to $(\mathrm{id}_{\mathcal{C}(U)_{w=0}}, 0)$ under the equivalence $\mathcal{C}(X)^u_{w=0} \xrightarrow{\sim} \mathcal{C}(U)_{w=0} \times \mathcal{C}(Z)^u_{w=0}$ of Theorem 2.9 (a).

Remark 2.11. According to the one of the main results of [AK], the categorial version of Wedderburn's theorem holds [AK, Thm. 12.1.1]. Thus, assuming that $\mathcal{C}(Z)_{w=0}$ and $\mathcal{C}(U)_{w=0}$ are Wedderburn categories (hence so is $\mathcal{C}(X)_{w=0}$), there exist sections

$$\iota : \overline{\mathcal{C}(X)_{w=0}} \hookrightarrow \mathcal{C}(X)_{w=0}$$

of the projection $pr_{\mathcal{C}(X)_{w=0}}$ from $\mathcal{C}(X)_{w=0}$ to $\overline{\mathcal{C}(X)_{w=0}}$. In an earlier approach, we defined an intermediate extension to be a functor

$$j_{!*} : \mathcal{C}(U)_{w=0} \longrightarrow \mathcal{C}(X)_{w=0}$$

of the form $\iota \circ \overline{j}_{!*} \circ pr_{\mathcal{C}(U)_{w=0}}$

$$\begin{CD}
\mathcal{C}(U)_{w=0} @>{j_{!*}}>> \mathcal{C}(X)_{w=0} \\
@V{pr_{\mathcal{C}(U)_{w=0}}}VV @AA{\iota}A \\
\overline{\mathcal{C}(U)_{w=0}} @>{\overline{j}_{!*}}>> \overline{\mathcal{C}(X)_{w=0}}
\end{CD}$$

where ι is a choice of such a section, and $\overline{j}_{!*}$ is defined as in Definition 2.10. The price to pay for a functor with target $\mathcal{C}(X)_{w=0}$ is that the relation $j^* \circ j_{!*} \cong \mathrm{id}_{\mathcal{C}(U)_{w=0}}$ does not necessarily hold; *a priori*, it is true only up to elements in the radical of $\mathcal{C}(U)_{w=0}$.

Let us conclude the section by listing the properties of $j_{!*}$.

Summary 2.12. *Let the F-linear pseudo-Abelian triangulated categories $\mathcal{C}(U)$, $\mathcal{C}(X)$ and $\mathcal{C}(Z)$ be related by gluing, and equipped with weight structures w compatible with the gluing. Assume $\mathcal{C}(Z)_{w=0}$ to be semi-primary.*

(a) Let M_U be an object of $\mathcal{C}(U)_{w=0}$. Then $j_{!*}M_U$ extends M_U, i.e., *$j^*j_{!*}M_U \cong M_U$. Furthermore, $j_{!*}M_U$ satisfies the following properties, any of which characterizes $j_{!*}M_U$ among the objects M of $\mathcal{C}(X)_{w=0}$ extending M_U, up to an isomorphism, which becomes unique in $\mathcal{C}(X)^u_{w=0}$ (and hence in $\overline{\mathcal{C}(X)_{w=0}}$).*

(1) $j_{!*}M_U$ *admits no non-zero direct factor belonging to* $i_*\mathcal{C}(Z)_{w=0}$.

(2a) For all objects N *of* $\mathcal{C}(Z)_{w=0}$,

$$\mathrm{Hom}_{\mathcal{C}(X)_{w=0}}(j_{!*}M_U, i_*N) = \mathrm{rad}_{\mathcal{C}(X)_{w=0}}(j_{!*}M_U, i_*N) \ .$$

(2b) For all objects N *of* $\mathcal{C}(Z)_{w=0}$,

$$\mathrm{Hom}_{\mathcal{C}(X)_{w=0}}(i_*N, j_{!*}M_U) = \mathrm{rad}_{\mathcal{C}(X)_{w=0}}(i_*N, j_{!*}M_U) \ .$$

(3a) The composition of the two adjunction morphisms

$$i_*i^!j_{!*}M_U \longrightarrow j_{!*}M_U \longrightarrow i_*i^*j_{!*}M_U$$

belongs to the radical

$$\mathrm{rad}_{\mathcal{C}(X)}(i_*i^!j_{!*}M_U, i_*i^*j_{!*}M_U) = \mathrm{rad}_{\mathcal{C}(Z)}(i^!j_{!*}M_U, i^*j_{!*}M_U) \ .$$

(3b) There exist weight filtrations

$$N_{\le -1} \longrightarrow i^*j_{!*}M_U \xrightarrow{m_+} N_0 \longrightarrow N_{\le -1}[1]$$

of $i^*j_{!*}M_U$ *(with* $N_{\le -1} \in \mathcal{C}(Z)_{w\le -1}$ *and* $N_0 \in \mathcal{C}(Z)_{w=0}$*), and*

$$L_0 \xrightarrow{m_-} i^!j_{!*}M_U \longrightarrow L_{\ge 1} \longrightarrow L_0[1]$$

of $i^!j_{!*}M_U$ *(with* $L_0 \in \mathcal{C}(Z)_{w=0}$ *and* $L_{\ge 1} \in \mathcal{C}(Z)_{w\ge 1}$*), such that the composition*

$$f : i_*L_0 \xrightarrow{i_*m_-} i_*i^!j_{!*}M_U \longrightarrow i_*i^*j_{!*}M_U \xrightarrow{i_*m_+} i_*N_0$$

of i_*m_-*, of the composition of the adjunction morphisms, and of* i_*m_+ *belongs to the radical:*

$$f \in \mathrm{rad}_{\mathcal{C}(X)_{w=0}}(i_*L_0, i_*N_0) = \mathrm{rad}_{\mathcal{C}(Z)_{w=0}}(L_0, N_0) \ .$$

(3c) For all weight filtrations

$$N_{\le -1} \longrightarrow i^*j_{!*}M_U \xrightarrow{m_+} N_0 \longrightarrow N_{\le -1}[1]$$

of $i^*j_{!*}M_U$*, and*

$$L_0 \xrightarrow{m_-} i^!j_{!*}M_U \longrightarrow L_{\ge 1} \longrightarrow L_0[1]$$

of $i^!j_{!*}M_U$*, the composition*

$$f : i_*L_0 \xrightarrow{i_*m_-} i_*i^!j_{!*}M_U \longrightarrow i_*i^*j_{!*}M_U \xrightarrow{i_*m_+} i_*N_0$$

belongs to the radical:

$$f \in \mathrm{rad}_{\mathcal{C}(X)_{w=0}}(i_*L_0, i_*N_0) = \mathrm{rad}_{\mathcal{C}(Z)_{w=0}}(L_0, N_0) \ .$$

(4a) The kernel of

$$j^* : \mathrm{End}_{\mathcal{C}(X)_{w=0}}(j_{!*}M_U) \longrightarrow\!\!\!\!\to \mathrm{End}_{\mathcal{C}(U)_{w=0}}(M_U)$$

is contained in $\mathrm{rad}_{\mathcal{C}(X)_{w=0}}(j_{!*}M_U, j_{!*}M_U)$.

(4b) Any element of the kernel of

$$j^* : \mathrm{End}_{\mathcal{C}(X)_{w=0}}(j_{!*}M_U) \longrightarrow\!\!\!\!\rightarrow \mathrm{End}_{\mathcal{C}(U)_{w=0}}(M_U)$$

is nilpotent.

(b) Any object M of $\mathcal{C}(X)_{w=0}$ is isomorphic to a direct sum $j_{!*}M_U \oplus i_*N$, for an object M_U of $\mathcal{C}(U)_{w=0}$ and an object N of $\mathcal{C}(Z)_{w=0}$. M_U is unique up to unique isomorphism; in fact, $M_U \cong j^*M$. N is unique up to an isomorphism, which becomes unique in $\mathcal{C}(Z)^u_{w=0}$ (and hence in $\overline{\mathcal{C}(Z)_{w=0}}$).
(c) The indecomposable objects of $\mathcal{C}(X)_{w=0}$ are precisely those of the form

$$j_{!*}M_U \ ,$$

for an indecomposable object M_U of $\mathcal{C}(U)_{w=0}$, or of the form

$$i_*N \ ,$$

for an indecomposable object N of $\mathcal{C}(Z)_{w=0}$.
(d) Let $f_U : M'_U \to M_U$ be a morphism in $\mathcal{C}(U)_{w=0}$. Then any extension of f_U to a morphism

$$f : j_{!*}M'_U \longrightarrow j_{!*}M_U$$

in $\mathcal{C}(X)_{w=0}$, i.e., *satisfying $j^*f = f_U$, maps to $j_{!*}f_U$ in the quotient category $\mathcal{C}(X)^u_{w=0}$.*
(e) Let f_U be an idempotent endomorphism of an object of $\mathcal{C}(U)_{w=0}$. Then $j_{!*}f_U \in \mathcal{C}(X)^u_{w=0}$ can be lifted idempotently to $\mathcal{C}(X)_{w=0}$.

Proof. By definition, property (1) of part (a) characterizes and is satisfied by $j_{!*}M_U$.

Part (b) follows from Theorem 1.9 (a), part (d) from the definitions, and part (e) from Theorem 1.9 (b).

Part (c) is implied by (b) and property (1) of (a).

The object $j_{!*}M_U$ satisfies properties (2a)–(4b) of (a): use Theorem 1.9 (b) and Lemma 2.7 (see Remark 2.8). Now let M be an object of $\mathcal{C}(X)_{w=0}$ such that $j^*M \cong M_U$. According to (b), M is isomorphic to a direct sum $j_{!*}M_U \oplus i_*N$, for an object N of $\mathcal{C}(Z)_{w=0}$. It is then easy to see that none of properties (2a)–(4b) holds when $j_{!*}M_U$ is replaced by M, unless $N = 0$.

q.e.d.

Example 2.13. Let F be field contained in $\mathbb{R}$, and X a scheme over $\mathbb{C}$. Put $\mathcal{C}(X) := D^b(\mathbf{MHM}_F X)$ as in Example 2.3 (b). As explained there, the additional presence of the t-structure ensures that $D^b(\mathbf{MHM}_F X)_{w=0}$ is a semi-primary category. Since $D^b(\mathbf{MHM}_F X)$ is pseudo-Abelian, the intermediate extension

$$j_{!*} : D^b(\mathbf{MHM}_F U)_{w=0} \longrightarrow D^b(\mathbf{MHM}_F X)^u_{w=0}$$

can be defined for any open sub-scheme $U \subset X$. Let us clarify its relation with the intermediate extension defined as the image of the canonical

transformation from $H^0 j_!$ to $H^0 j_*$ from [Sa2, (4.5.10)], which we shall denote by

$$j_{!*}^{t=0} : \mathbf{MHM}_F\, U \longrightarrow \mathbf{MHM}_F\, X \; .$$

Let $M_U \in \mathbf{MHM}_F\, U$ be pure of weight $n \in \mathbb{Z}$. Then $M_U[-n]$ belongs to $D^b(\mathbf{MHM}_F\, U)_{w=0}$. According to [Sa2, (4.5.2)],

$$(j_{!*}^{t=0} M_U)[-n] \in D^b(\mathbf{MHM}_F\, X)_{w=0} \; .$$

Now the composition of the adjunction morphisms

$$i_* i^! (j_{!*}^{t=0} M_U) \longrightarrow j_{!*}^{t=0} M_U \longrightarrow i_* i^* (j_{!*}^{t=0} M_U)$$

belongs to the radical (since the source is in $D^b(\mathbf{MHM}_F\, X)^{t \ge 1}$, and the target in $D^b(\mathbf{MHM}_F\, X)^{t \le -1}$). According to Summary 2.12 (a) (3),

$$(j_{!*}^{t=0} M_U)[-n] \cong j_{!*}(M_U[-n]) \; .$$

In fact, the isomorphism is unique once we require it to give the identity on U: recall that $D^b(\mathbf{MHM}_F\, X)_{w=0}^{t=n}$ is Abelian semi-simple, therefore, the radical $\mathrm{rad}_{D^b(\mathbf{MHM}_F\, X)}(j_{!*}(M_U[-n], j_{!*}(M_U[-n])$ is trivial. We get

$$[-n] \circ j_{!*}^{t=0} \circ [n] = j_{!*} \quad \text{on} \quad D^b(\mathbf{MHM}_F\, X)_{w=0}^{t=n} \; ,$$

canonically for any integer n.

Remark 2.14. Example 2.13 illustrates that the intermediate extension behaves best on objects which are concentrated simultaneously in a single t-degree and in a single w-degree. The absence of control of the t-degree leads to choices in $\mathcal{C}(X)_{w=0}$, which in general become unique only in the quotient category $\mathcal{C}(X)_{w=0}^u$. The absence of control of the w-degree leads to non-exactness of $j_{!*}^{t=0}$. Historically, the second phenomenon seems to be well tolerated (actually, the author knows of no successful application of $j_{!*}^{t=0}$ to objects which are not pure...). We think of the first as its weight-structural counterpart.

Remark 2.15. (a) The intermediate extension behaves well under passage to opposite categories. This is true first for its underlying notions. Examples: an F-linear category $\mathfrak{B}$ is semi-primary if and only if the opposite category $\mathfrak{B}^{opp}$ is; the triangulated categories $\mathcal{C}(U)$, $\mathcal{C}(X)$ and $\mathcal{C}(Z)$ are related by gluing if and only if the opposite categories $\mathcal{C}(U)^{opp}$, $\mathcal{C}(X)^{opp}$ and $\mathcal{C}(Z)^{opp}$ are — here of course, we have to make the obvious modifications on the level of the six exact functors used to glue (*e.g.*, the functor $j_*^{opp} : \mathcal{C}(U)^{opp} \to \mathcal{C}(X)^{opp}$ is opposite to $j_!$). Similarly, these categories carry weight structures w compatible with the gluing if and only if the opposite categories do — here, in order to obtain the weight structure w^{opp} opposite to w, we have to invert the sign of the weight (*e.g.*, the category $\mathcal{C}(X)^{opp}_{w^{opp} \le 0}$ is opposite to $\mathcal{C}(X)_{w \ge 0}$). In particular,

$$\mathcal{C}(X)^{opp}_{w^{opp}=0} = \big(\mathcal{C}(X)_{w=0}\big)^{opp} \; ,$$

and similarly for U and Z instead of X. Thus, the functor

$$j_{!*}^{opp} : \big(\mathcal{C}(U)_{w=0}\big)^{opp} \hookrightarrow \big(\mathcal{C}(X)^u_{w=0}\big)^{opp}$$

is defined as in Definition 2.10. One notes then that $j_{!*}^{opp}$ is in fact the functor opposite to $j_{!*}$.
(b) The intermediate extension is compatible with duality. More precisely, assume that equivalences

$$\mathbb{D}_{\bullet} : \mathcal{C}(\bullet) \xrightarrow{\sim} \mathcal{C}(\bullet)^{opp} , \ \bullet \in \{U, X, Z\}$$

are given, that they are compatible with the gluing (*e.g.*, $\mathbb{D}_X \circ j_* \cong j_*^{opp} \circ \mathbb{D}_U$) and with the weight structures (*e.g.*, $\mathbb{D}_X \mathcal{C}(X)_{w\ge 0} = \mathcal{C}(X)^{opp}_{w^{opp}\ge 0}$). It then follows formally from (a) that

$$\mathbb{D}_X \circ j_{!*} \cong j_{!*}^{opp} \circ \mathbb{D}_U \ .$$

One can study compatibility of intermediate extensions with other types of operations, for example direct and inverse images (see Section 6). In that context, the following notion turns out to be essential.

Definition 2.16 ([AK, Déf. 1.4.6])**.** An F-linear functor $r : \mathfrak{A}_1 \to \mathfrak{A}_2$ between F-linear categories is called *radicial* if it maps the radical of $\mathfrak{A}_1$ to the radical of $\mathfrak{A}_2$:

$$r(\mathrm{rad}_{\mathfrak{A}_1}) \subset \mathrm{rad}_{\mathfrak{A}_2} \ .$$

Proposition 2.17. *Assume that two triples of F-linear pseudo-Abelian triangulated categories, related by gluing, and equipped with weight structures w compatible with the gluing are given: $\mathcal{C}_1(U)$, $\mathcal{C}_1(X)$, $\mathcal{C}_1(Z)$, and $\mathcal{C}_2(U)$, $\mathcal{C}_2(X)$, $\mathcal{C}_2(Z)$ (we use the same symbols j^*, i_** etc. *for the two sets of gluing functors). Assume that furthermore, F-linear exact functors*

$$r_U : \mathcal{C}_1(U) \longrightarrow \mathcal{C}_2(U) \ , \ r_X : \mathcal{C}_1(X) \longrightarrow \mathcal{C}_2(X) \ , \ r_Z : \mathcal{C}_1(Z) \longrightarrow \mathcal{C}_2(Z)$$

*are given, which commute with the gluing functors, and which are w-*exact*:*

$$r_{\bullet}\big(\mathcal{C}_1(\bullet)_{w\le 0}\big) \subset \mathcal{C}_2(\bullet)_{w\le 0} \quad \text{and} \quad r_{\bullet}\big(\mathcal{C}_1(\bullet)_{w\ge 0}\big) \subset \mathcal{C}_2(\bullet)_{w\ge 0} \ .$$

In particular, they induce functors $r_{\bullet} : \mathcal{C}_1(\bullet)_{w=0} \to \mathcal{C}_2(\bullet)_{w=0}$. Assume that $\mathcal{C}_1(Z)_{w=0}$ and $\mathcal{C}_2(Z)_{w=0}$ are semi-primary.
(a) If $r_Z : \mathcal{C}_1(Z)_{w=0} \to \mathcal{C}_2(Z)_{w=0}$ is radicial, then r_X maps any object M of $\mathcal{C}_1(X)_{w=0}$ without non-zero direct factors in $i_*\mathcal{C}_1(Z)_{w=0}$ to an object $r_X M$ of $\mathcal{C}_2(X)_{w=0}$ without non-zero direct factors in $i_*\mathcal{C}_2(Z)_{w=0}$. The functor r_X descends to induce a functor

$$r_X^u : \mathcal{C}_1(X)^u_{w=0} \longrightarrow \mathcal{C}_2(X)^u_{w=0} \ ,$$

and r_X^u respects the decompositions

$$\mathcal{C}_m(X)^u_{w=0} \xrightarrow{\sim} \mathcal{C}_m(U)_{w=0} \times \mathcal{C}_m(Z)^u_{w=0} \ ,$$

$m = 1, 2$, *from Theorem 2.9 (a). In particular, the diagram*

$$\begin{tikzcd} \mathcal{C}_1(U)_{w=0} \arrow[r, hook, "j_{!*}"] \arrow[d, "r_U"'] & \mathcal{C}_1(X)^u_{w=0} \arrow[d, "r^u_X"'] \\ \mathcal{C}_2(U)_{w=0} \arrow[r, hook, "j_{!*}"] & \mathcal{C}_2(X)^u_{w=0} \end{tikzcd}$$

commutes.

(b) If both $r_Z : \mathcal{C}_1(Z)_{w=0} \to \mathcal{C}_2(Z)_{w=0}$ *and* $r_U : \mathcal{C}_1(U)_{w=0} \to \mathcal{C}_2(U)_{w=0}$ *are radicial, then so is* $r_X : \mathcal{C}_1(X)_{w=0} \to \mathcal{C}_2(X)_{w=0}$. *The functor* r^u_X *descends further to induce a functor*

$$\overline{r_X} : \overline{\mathcal{C}_1(X)_{w=0}} \longrightarrow \overline{\mathcal{C}_2(X)_{w=0}} \ ,$$

and $\overline{r_X}$ *respects the decompositions*

$$\overline{\mathcal{C}_m(X)_{w=0}} \xrightarrow{\ \sim\ } \overline{\mathcal{C}_m(U)_{w=0}} \times \overline{\mathcal{C}_m(Z)_{w=0}} \ ,$$

$m = 1, 2$, *from Theorem 2.9 (a).*

Proof. (a): Since M is supposed to satisfy condition 2.12 (a) (1), it satisfies the equivalent condition 2.12 (a) (3b). The latter concerns a morphism belonging to the radical of $\mathcal{C}_1(Z)_{w=0}$. Thanks to our assumption on the restriction of r_Z to $\mathcal{C}_1(Z)_{w=0}$, condition 2.12 (a) (3b) is then equally satisfied by $r_X M$. Therefore, condition 2.12 (a) (1) is also fulfilled, which proves the claim concerning $r_X M$. The claims concerning r^u_X then follow from Definition 1.6.

(b): Theorem 2.9 (a) allows us to assume that the radicals of $\mathcal{C}_2(\bullet)_{w=0}$ are all trivial, and that we are in a trivial gluing situation:

$$\mathcal{C}_2(X)_{w=0} = \mathcal{C}_2(U)_{w=0} \times \mathcal{C}_2(Z)_{w=0} \ .$$

The assumption is that r_Z and r_U are trivial on the radicals of $\mathcal{C}_1(Z)_{w=0}$ and $\mathcal{C}_1(U)_{w=0}$, respectively. Let M and N be objects of $\mathcal{C}_1(X)_{w=0}$, and

$$f : M \longrightarrow N$$

a morphism belonging to the radical. We need to show that $r_X f = 0$. According to Theorem 2.9 (a), we may assume that M belongs to the image of $j_{!*}$ (meaning that it does not admit non-zero direct factors in the image of i_*) or to the image of i_*, and likewise for N. If both M and N are in the image of $j_{!*}$, then according to (a), so are $r_X M$ and $r_X N$. Furthermore, according to our assumption, $j^* r_X f = 0$. But then $r_X f = 0$. One argues similarly if M and N are both in the image of i_*. If one of them is on the image of $j_{!*}$ and the other in the image of i_*, then there are no non-trivial morphisms in $\mathcal{C}_2(X)_{w=0}$ between their images under r_X. **q.e.d.**

3 The motivic picture

From now on, F is assumed to be a finite direct product of fields of characteristic zero. In addition, let us fix a base scheme $\mathbb{B}$, which is of finite type over an excellent scheme of dimension at most two. The conventions on schemes, morphisms and regularity are those fixed in our Introduction. We use the triangulated, $\mathbb{Q}$-linear categories $DM_{\mathrm{B},c}(X)$ of *constructible Beilinson motives* over X [CD, Def. 15.1.1], indexed by schemes X (always in the sense of the above conventions). In order to have an F-linear theory at one's disposal, one re-does the construction, but using F instead of $\mathbb{Q}$ as coefficients [CD, Sect. 15.2.5]. This yields triangulated, F-linear categories $DM_{\mathrm{B},c}(X)_F$ satisfying the F-linear analogues of the properties of $DM_{\mathrm{B},c}(X)$. In particular, these categories are pseudo-Abelian (see [H, Sect. 2.10]). Furthermore, the canonical functor $DM_{\mathrm{B},c}(X) \otimes_{\mathbb{Q}} F \to DM_{\mathrm{B},c}(X)_F$ is fully faithful [CD, Sect. 14.2.20]. As in [CD], the symbol $\mathbb{1}_X$ is used to denote the unit for the tensor product in $DM_{\mathrm{B},c}(X)_F$. We shall employ the full formalism of six operations developed in [loc. cit.]. The reader may choose to consult [H, Sect. 2] or [W4, Sect. 1] for concise presentations of this formalism.

Beilinson motives can be endowed with weight structures, thanks to the main results from [H] (see [B1, Prop. 6.5.3] for the case $X = \mathbf{Spec}\, k$, for a perfect field k). More precisely, the following holds.

Theorem 3.1 ([H, Thm. 3.3 (ii), Thm. 3.8 (i)–(ii), Rem. 3.4])**.** *There are canonical weight structures w on the categories $DM_{\mathrm{B},c}(\bullet)$. They have the following properties.*

(1) The objects $\mathbb{1}_X(p)[2p]$ belong to the heart $DM_{\mathrm{B},c}(X)_{w=0}$, for all integers p, whenever X is regular.

(2) For a morphism of schemes f, left adjoint functors f^*, $f_!$ respect the full sub-categories $DM_{\mathrm{B},c}(\bullet)_{w\le 0}$, and right adjoint functors f_*, $f^!$ and f^* (the latter for smooth f) respect $DM_{\mathrm{B},c}(\bullet)_{w\ge 0}$.

(3) For a fixed scheme X, the heart of w is the pseudo-Abelian completion of the category of motives over X of the form

$$f_*\mathbb{1}_S(p)[2p]\ ,$$

for proper morphisms $f : S \to X$ with regular source S, and integers p. The same statement holds, with "proper" replaced by "projective".

Furthermore, properties (1) and (2) together characterize the weight structures w uniquely.

Since by [CD, Prop. 15.2.3], motives of type 3.1 (3) generate $DM_{\mathrm{B},c}(X)$ as a thick triangulated category, we see in particular that the latter is generated by the heart of its weight structure. For that reason, the proof of

[H, Thm. 3.3, Thm. 3.8] can be imitated to show that the analogue of Theorem 3.1 holds for the F-linear versions $DM_{\text{Б},c}(\bullet)_F$ of the categories $DM_{\text{Б},c}(\bullet)$. Let us refer to the weight structure w on $DM_{\text{Б},c}(\bullet)_F$ as the *motivic weight structure.*

Definition 3.2 (cmp. [W4, Def. 1.5])**.** The category $CHM(X)_F$ of *Chow motives* over X is defined as the heart $DM_{\text{Б},c}(X)_{F,w=0}$ of the motivic weight structure on $DM_{\text{Б},c}(X)_F$.

Thus, the category $CHM(X)_F$ equals the pseudo-Abelian completion of $CHM(X)_{\mathbb{Q}} \otimes_{\mathbb{Q}} F$.

If $i : Z \hookrightarrow X$ and $j : U \hookrightarrow X$ are complementary closed, resp. open immersions of schemes, then the category $DM_{\text{Б},c}(X)_F$ is obtained by gluing $DM_{\text{Б},c}(U)_F$ and $DM_{\text{Б},c}(Z)_F$ *via* the functors

$$DM_{\text{Б},c}(U)_F \xrightarrow{j_!} DM_{\text{Б},c}(X)_F \xrightarrow{i^*} DM_{\text{Б},c}(Z)_F$$

$$DM_{\text{Б},c}(U)_F \xleftarrow{j^*} DM_{\text{Б},c}(X)_F \xleftarrow{i_*} DM_{\text{Б},c}(Z)_F$$

$$DM_{\text{Б},c}(U)_F \xrightarrow{j_*} DM_{\text{Б},c}(X)_F \xrightarrow{i^!} DM_{\text{Б},c}(Z)_F$$

[CD, Prop. 2.3.3 (2), (3), Thm. 2.2.14 (3), Sect. 2.3.1]. We are therefore in the situation studied in Section 2.

Even though the results of this paper are unconditional, it may be useful to indicate that our global vision is guided by the following.

Conjecture 3.3. *Let X be a scheme, and M an indecomposable object of $CHM(X)_F$. Then the radical*

$$\mathrm{rad}_{CHM(X)_F}(M, M) \subset \mathrm{End}_{CHM(X)_F}(M)$$

is trivial.

Conjecture 3.4. *Let X be a scheme. Then the category $CHM(X)_F$ is semi-primary.*

When X is the spectrum of a field k, then Conjecture 3.4 is a consequence of a conjecture proposed independently by Kimura and O'Sullivan (*c.f.* [Ad, Sect. 12.1.2]). It predicts that all objects of $CHM(\mathbf{Spec}\, k)_F$ are *finite dimensional*. This notion will turn out to be very important for us; its precise definition will be recalled later (Definition 5.3). For the moment, let us note that according to [AK, Thm. 9.2.2] (see also [O'S1, Lemma 4.1]), any pseudo-Abelian F-linear symmetric rigid tensor category is (Wedderburn, hence) semi-primary, as soon as all of its objects are finite dimensional.

Conjectures 3.3 and 3.4 would be immediate consequences of the existence of a t-structure on $DM_{\mathrm{B},c}(X)_F$, which is transversal to the motivic weight structure (see Example 2.3 (a)). Conjecture 3.4 guarantees the existence of the intermediate extension

$$j_{!*} : CHM(U)_F \longrightarrow CHM(X)_F^u .$$

If M_U is an indecomposable object of $CHM(U)_F$, then according to Summary 2.12 (c), $j_{!*}M_U$ is indecomposable in $CHM(X)_F$. Conjecture 3.3 then predicts the triviality of its radical, hence the projection

$$\mathrm{End}_{CHM(X)_F}(j_{!*}M_U) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{End}_{CHM(X)_F^u}(j_{!*}M_U)$$

would be an isomorphism. In particular, any endomorphism h_U of M_U would then admit a unique extension to an endomorphism of $j_{!*}M_U$ in $CHM(X)_F$.

Example 3.5. Assume U to be regular, and dense in X. According to Theorem 3.1 (1), the object $\mathbb{1}_U$ then belongs to $CHM(U)_F$. On the one hand, Conjecture 3.4 would allow to define the value of intermediate extension on $\mathbb{1}_U$,

$$j_{!*}\mathbb{1}_U \in CHM(X)_F^u .$$

On the other hand, the *motivic intersection complex*, denoted by the same symbol $j_{!*}\mathbb{1}_U$, is defined as an extension of $\mathbb{1}_U$ to a Chow motive over X, satisfying

$$j^* : \mathrm{End}_{CHM(X)_F}(j_{!*}\mathbb{1}_U) \xrightarrow{\ \sim\ } \mathrm{End}_{CHM(U)_F}(\mathbb{1}_U)$$

[W4, Def. 2.1]. If it exists, then Conjecture 3.3 holds for $M = j_{!*}\mathbb{1}_U$, since $\mathrm{End}_{CHM(X)_F}(j_{!*}\mathbb{1}_U)$ is then equal to a finite product of copies of F. By the above discussion, the two notions are compatible: the motivic intersection complex would thus equal the value of the functor $j_{!*}$ on $\mathbb{1}_U$.

The main issue to be treated in Sections 4 and 5 is the construction of full sub-categories of Chow motives, which can actually be shown to satisfy Conjecture 3.4, *i.e.*, to be semi-primary.

4 Gluing of motives

The construction we aim at relies on the process of gluing certain "nice" sub-categories of motives over the members of a stratification. The following abstract criterion tells us when it is actually possible to glue given sub-categories.

Proposition 4.1. *Let the three F-linear triangulated categories $\mathcal{C}(U)$, $\mathcal{C}(X)$ and $\mathcal{C}(Z)$ be related by gluing:*

$$\mathcal{C}(U) \xrightarrow{j_!} \mathcal{C}(X) \xrightarrow{i^*} \mathcal{C}(Z)$$

$$\mathcal{C}(U) \xleftarrow{j^*} \mathcal{C}(X) \xleftarrow{i_*} \mathcal{C}(Z)$$

$$\mathcal{C}(U) \xrightarrow{j_*} \mathcal{C}(X) \xrightarrow{i^!} \mathcal{C}(Z)$$

Let $\mathcal{D}(U) \subset \mathcal{C}(U)$ and $\mathcal{D}(Z) \subset \mathcal{C}(Z)$ be full, triangulated sub-categories.
(a) $\mathcal{D}(U)$ and $\mathcal{D}(Z)$ can be glued via *$(j_!, j^*, j_*, i^*, i_*, i^!)$ to give a full, triangulated sub-category $\mathcal{D}(X)$ of $\mathcal{C}(X)$ if and only if for all objects M_U of $\mathcal{D}(U)$, both $i^*j_*M_U$ and $i^!j_!M_U$ belong to $\mathcal{D}(Z)$ (equivalently: $i^*j_*M_U$* or *$i^!j_!M_U$ belongs to $\mathcal{D}(Z)$).*
(b) Assume that the condition from (a) is satisfied. Let $M \in \mathcal{C}(X)$, and assume that $j^*M \in \mathcal{D}(U)$. Then the following conditions are equivalent.

(1) $M \in \mathcal{D}(X)$.

(2) $i^*M \in \mathcal{D}(Z)$.

(3) $i^!M \in \mathcal{D}(Z)$.

Proof. Note that by [BBD, Formula (1.4.6.4)], the functors i^*j_* and $i^!j_![1]$ are isomorphic. The category $\mathcal{D}(Z)$ is triangulated, hence the condition $i^*j_*M_U \in \mathcal{D}(Z)$ is equivalent to $i^!j_!M_U \in \mathcal{D}(Z)$. Given this observation, the proof of part (a) is straightforward, as is the description of $\mathcal{D}(X)$: an object M of $\mathcal{C}(X)$ belongs to $\mathcal{D}(X)$ if and only if $j^*M \in \mathcal{D}(U)$, $i^*M \in \mathcal{D}(Z)$ and $i^!M \in \mathcal{D}(Z)$.

In order to prove part (b), it thus suffices to establish that claim (1) is implied by both claim (2) and claim (3). In order to show "(2) $\Rightarrow$ (1)", let $M \in \mathcal{C}(X)$, and assume that $j^*M \in \mathcal{D}(U)$ and $i^*M \in \mathcal{D}(Z)$. We have to show that $i^!M \in \mathcal{D}(Z)$. In order to do so, consider the localization triangle

$$j_!j^*M \longrightarrow M \longrightarrow i_*i^*M \longrightarrow j_!j^*M[1]$$

[BBD, Axiom (1.4.3.4)], and apply $i^!$, to get

$$i^!j_!j^*M \longrightarrow i^!M \longrightarrow i^*M \longrightarrow i^!j_!j^*M[1] \ .$$

Since $j^*M \in \mathcal{D}(U)$ and the condition from (a) is satisfied, the term $i^!j_!j^*M$ belongs to $\mathcal{D}(Z)$. The same is true for i^*M. Therefore, $i^!M \in \mathcal{D}(Z)$. The proof of "(3) $\Rightarrow$ (1)" is dual. **q.e.d.**

Remark 4.2. Assume that $\mathcal{D}(U) \subset \mathcal{C}(U)$ and $\mathcal{D}(Z) \subset \mathcal{C}(Z)$ can be glued to give $\mathcal{D}(X) \subset \mathcal{C}(X)$.
(a) It follows from the localization triangle that $\mathcal{D}(X)$ can be described as

the full, triangulated sub-category of $\mathcal{C}(X)$ generated by $j_!\mathcal{D}(U)$ and $i_*\mathcal{D}(Z)$. Dually, it equals the full, triangulated sub-category of $\mathcal{C}(X)$ generated by $j_*\mathcal{D}(U)$ and $i_*\mathcal{D}(Z)$.
(b) The sub-category $\mathcal{D}(X)$ is dense in $\mathcal{C}(X)$ if and only $\mathcal{D}(U)$ and $\mathcal{D}(Z)$ are dense in $\mathcal{C}(U)$ and $\mathcal{C}(Z)$, respectively.

Let us apply the above to motives. Let S be a scheme. By definition, a *good stratification* of S indexed by a finite set $\mathfrak{S}$ is a collection of locally closed sub-schemes S_σ indexed by $\sigma \in \mathfrak{S}$, such that

$$S = \coprod_{\sigma \in \mathfrak{S}} S_\sigma$$

on the set-theoretic level, and such that the closure $\overline{S_\sigma}$ of any stratum S_σ is a union of strata S_τ. In that situation, we shall often write $S(\mathfrak{S})$ instead of S.

Our first concrete example concerns the category built up from Tate motives.

Definition 4.3 (cmp. [Le1, Def. 3.1], [Le3, Sect. 3.3])**.** Let X be a scheme. Define the *category of Tate motives over* X as the strict, full, triangulated sub-category $DMT(X)_F$ of $DM_{\mathrm{B},c}(X)_F$ generated by the $\mathbb{1}_X(p)$, for $p \in \mathbb{Z}$.

Recall that by definition, a strict sub-category is closed under isomorphisms in the ambient category.

Theorem 4.4. *Let $S(\mathfrak{S}) = \coprod_{\sigma \in \mathfrak{S}} S_\sigma$ be a good stratification of a scheme $S(\mathfrak{S})$. Assume that the closures $\overline{S_\sigma}$ of all strata S_σ, $\sigma \in \mathfrak{S}$ are regular.*
(a) The categories of Tate motives $DMT(S_\sigma)_F$, $\sigma \in \mathfrak{S}$, can be glued to give a full, triangulated sub-category $DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F$ of $DM_{\mathrm{B},c}(S(\mathfrak{S}))_F$.
(b) Let $M \in DM_{\mathrm{B},c}(S(\mathfrak{S}))_F$. Then the following conditions are equivalent.

(1) $M \in DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F$.

(2) $j^*M \in DMT(S_\sigma)_F$, for all $\sigma \in \mathfrak{S}$, where j denotes the immersion $S_\sigma \hookrightarrow S(\mathfrak{S})$.

(3) $j^!M \in DMT(S_\sigma)_F$, for all $\sigma \in \mathfrak{S}$.

In particular, the triangulated category $DMT(S(\mathfrak{S}))_F$ of Tate motives over $S(\mathfrak{S})$ is contained in $DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F$.
(c) The category $DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F$ is the strict, full, triangulated sub-category of $DM_{\mathrm{B},c}(S(\mathfrak{S}))_F$ generated by the $\overline{j}_* \mathbb{1}_{\overline{S_\sigma}}(p)$, for $\sigma \in \mathfrak{S}$ and $p \in \mathbb{Z}$, where $\overline{j}$ denotes the closed immersion $\overline{S_\sigma} \hookrightarrow S(\mathfrak{S})$.

Definition 4.5. Let $S(\mathfrak{S}) = \coprod_{\sigma \in \mathfrak{S}} S_\sigma$ be a good stratification of a scheme $S(\mathfrak{S})$, such that the closures $\overline{S_\sigma}$ of all strata S_σ, $\sigma \in \mathfrak{S}$ are regular.

(a) The category $DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F$ of Theorem 4.4 is called the *category of $\mathfrak{S}$-constructible Tate motives over $S(\mathfrak{S})$*.
(b) The full sub-category $CHMT_{\mathfrak{S}}(S(\mathfrak{S}))_F$ of $CHM(S(\mathfrak{S}))_F$ of objects lying in both $DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F$ and $CHM(S(\mathfrak{S}))_F$ is called the *category of $\mathfrak{S}$-constructible Chow–Tate motives over $S(\mathfrak{S})$*.

Example 4.6. Let K be a number field, and denote by S the spectrum of its ring of integers $\mathfrak{o}_K$. Let $\mathfrak{S}$ be a set consisting of a finite number of closed points of S (*i.e.*, of maximal ideals of $\mathfrak{o}_K$), and of the open complement of their union. Then $S = \coprod_{\sigma \in \mathfrak{S}} S_\sigma$ satisfies the hypothesis of Theorem 4.4 and Definition 4.5. Thus, the category $DMT_{\mathfrak{S}}(\mathbf{Spec}\,\mathfrak{o}_K)_F$ of $\mathfrak{S}$-constructible Tate motives over $\mathbf{Spec}\,\mathfrak{o}_K$ is defined, and so is the category $CHMT_{\mathfrak{S}}(\mathbf{Spec}\,\mathfrak{o}_K)_F$ of $\mathfrak{S}$-constructible Chow–Tate motives over $\mathbf{Spec}\,\mathfrak{o}_K$. By passing to the limit of $DMT_{\mathfrak{S}}(\mathbf{Spec}\,\mathfrak{o}_K)_F$ over all $\mathfrak{S}$, for $F = \mathbb{Q}$, one recovers the category denoted $\mathbf{DTM}(\mathbf{Spec}\,\mathfrak{o}_K)$ in [Slb, Def. 2.2]. Theorem 4.4 (b) and (c) continues to apply (with $\mathfrak{S}$ = the set of all points of $\mathbf{Spec}\,\mathfrak{o}_K$). In particular, we recover [Slb, Thm. 2.4].

Proof of Theorem 4.4. We proceed by induction on the number of strata, all claims being trivial if this number equals one.

Let S_σ be an open stratum; our induction hypothesis implies that Theorem 4.4 is true for any locally closed union of strata of the complement $S(\mathfrak{S}) - S_\sigma$, with its induced stratification.

Our aim is to glue $DMT(S_\sigma)_F$ and $DMT_{\mathfrak{S}}(S(\mathfrak{S}) - S_\sigma)_F$. By induction, we may assume that $S(\mathfrak{S}) = \overline{S_\sigma}$. Write j for the open immersion $S_\sigma \hookrightarrow \overline{S_\sigma}$, and i for the complementary closed immersion $\overline{S_\sigma} - S_\sigma \hookrightarrow \overline{S_\sigma}$.

Since Theorem 4.4 holds for $\overline{S_\sigma} - S_\sigma$, the motive $i^*\mathbb{1}_{\overline{S_\sigma}}(p) = \mathbb{1}_{\overline{S_\sigma} - S_\sigma}(p)$ belongs to $DMT_{\mathfrak{S}}(\overline{S_\sigma} - S_\sigma)_F$, for all $p \in \mathbb{Z}$. We claim that the same is true for the motive $i^!\mathbb{1}_{\overline{S_\sigma}}(p)$. Indeed, according to criterion (b) (3), applied to $\overline{S_\sigma} - S_\sigma$, it is sufficient to show that $i_\tau^!\mathbb{1}_{\overline{S_\sigma}}(p)$ is a Tate motive over S_τ, for the immersion $i_\tau : S_\tau \hookrightarrow \overline{S_\sigma}$ of any stratum S_τ of $\overline{S_\sigma} - S_\sigma$. This is where the hypothesis on $\overline{S_\sigma}$ enters: it is regular, and so is S_τ (since $\overline{S_\tau}$ is). *Absolute purity* [CD, Thm. 14.4.1] then implies that $i_\tau^!\mathbb{1}_{\overline{S_\sigma}}(p) \in DMT(S_\tau)_F$.

Now $i^*j_*\mathbb{1}_{S_\sigma}(p)$ is a cone of the canonical morphism $i^!\mathbb{1}_{\overline{S_\sigma}}(p) \to i^*\mathbb{1}_{\overline{S_\sigma}}(p)$ (see the proof of Proposition 4.1); from what precedes, we see that $i^*j_*\mathbb{1}_{S_\sigma}(p)$ belongs indeed to $DMT_{\mathfrak{S}}(\overline{S_\sigma} - S_\sigma)_F$. The composition i^*j_* thus maps $\mathbb{1}_{S_\sigma}(p)$ to an object of $DMT_{\mathfrak{S}}(\overline{S_\sigma} - S_\sigma)_F$.

Since i^*j_* maps the generators of $DMT(S_\sigma)_F$ to $DMT_{\mathfrak{S}}(\overline{S_\sigma} - S_\sigma)_F$, the whole of $DMT(S_\sigma)_F$ is in fact mapped to $DMT_{\mathfrak{S}}(\overline{S_\sigma} - S_\sigma)_F$ under i^*j_*. This shows that the criterion from Proposition 4.1 (a) is fulfilled, thereby proving part (a) of our claim.

Part (b) is Proposition 4.1 (b), applied to our situation.

By Remark 4.2 (a), $DMT_{\mathfrak{S}}(\overline{S_\sigma})_F$ is the strict, full, triangulated sub-category of $DM_{\text{Б},c}(\overline{S_\sigma})_F$ generated by the $j_!\mathbb{1}_{S_\sigma}(p)$, for all $p \in \mathbb{Z}$, and by the category $i_*DMT_{\mathfrak{S}}(\overline{S_\sigma} - S_\sigma)_F$. In order to establish (c), it thus remains

to show that modulo $i_* DMT_{\mathfrak{S}}(\overline{S_\sigma} - S_\sigma)_F$, the $j_! \mathbb{1}_{S_\sigma}(p)$ and the $\mathbb{1}_{\overline{S_\sigma}}(p)$ generate the same full, triangulated sub-category. But this follows from the localization triangle

$$j_! \mathbb{1}_{S_\sigma}(p) \longrightarrow \mathbb{1}_{\overline{S_\sigma}}(p) \longrightarrow i_* \mathbb{1}_{\overline{S_\sigma} - S_\sigma} \longrightarrow j_! j^* \mathbb{1}_{S_\sigma}(p)[1] \; .$$

q.e.d.

Remark 4.7. As its proof shows, Theorem 4.4 remains true when the assumption on regularity of $\overline{S_\sigma}$ is replaced by the following: for any $\sigma \in \mathfrak{S}$, and for any stratum $i_\tau : S_\tau \hookrightarrow \overline{S_\sigma}$ of $\overline{S_\sigma}$, the functor

$$i_\tau^! : DM_{\text{Б},c}(\overline{S_\sigma})_F \longrightarrow DM_{\text{Б},c}(S_\tau)_F$$

maps $\mathbb{1}_{\overline{S_\sigma}}$ to a Tate motive over S_τ.

As we shall see (Theorem 5.4), the category $CHMT_{\mathfrak{S}}(S(\mathfrak{S}))_F$ of $\mathfrak{S}$-constructible Chow–Tate motives over $S(\mathfrak{S})$ is (Wedderburn, hence) semi-primary, thus providing us with a non-trivial example of a sub-category of Chow motives for which the analogue of Conjecture 3.4 is true. Actually, we aim at a more general result; to obtain it, let us generalize the geometric setting.

Fix $S(\mathfrak{S}) = \coprod_{\sigma \in \mathfrak{S}} S_\sigma$, add a second good stratification $Y(\Phi) = \coprod_{\varphi \in \Phi} Y_\varphi$ of a scheme $Y(\Phi)$, and a morphism $\pi : S(\mathfrak{S}) \to Y(\Phi)$. We assume that the pre-image $\pi^{-1}(Y_\varphi)$ of any stratum Y_φ of $Y(\Phi)$ is a union of strata S_σ. To abbreviate, let us refer to π as a *morphism of good stratifications*.

Definition 4.8. Let $\pi : S(\mathfrak{S}) \to Y(\Phi)$ be a morphism of good stratifications. Assume that the closures $\overline{S_\sigma}$ of all strata S_σ, $\sigma \in \mathfrak{S}$ are regular. Define the category $\pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F$ as the strict, full, triangulated sub-category of $DM_{\text{Б},c}(Y(\Phi))_F$ generated by the images under π_* of the objects of $DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F$.

Using proper base change [CD, Thm. 2.4.50 (4)], one deduces the following from Theorem 4.4.

Corollary 4.9. *Let $\pi : S(\mathfrak{S}) \to Y(\Phi)$ be a proper morphism of good stratifications. Assume that the closures $\overline{S_\sigma}$ of all strata S_σ, $\sigma \in \mathfrak{S}$ are regular.*
(a) $\pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F$ is obtained by gluing the $\pi_* DMT_{\mathfrak{S}}(\pi^{-1}(Y_\varphi))_F$, $\varphi \in \Phi$.
(b) The category $\pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F$ is the strict, full, triangulated sub-category of $DM_{\text{Б},c}(Y(\Phi))_F$ generated by the $\pi_{\bar\sigma,*} \mathbb{1}_{\overline{S_\sigma}}(p)$, for $\sigma \in \mathfrak{S}$ and $p \in \mathbb{Z}$, where $\pi_{\bar\sigma}$ denotes the restriction of π to the closure $\overline{S_\sigma} \subset S(\mathfrak{S})$.

Corollary 4.10. *Let $\pi : S(\mathfrak{S}) \to Y(\Phi)$ be a proper morphism of good stratifications. Assume that the closures $\overline{S_\sigma}$ of all strata S_σ, $\sigma \in \mathfrak{S}$ are regular.*

(a) The motivic weight structure w on $DM_{\text{Б},c}(Y(\Phi))_F$ induces a weight structure, still denoted w, on $\pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F \subset DM_{\text{Б},c}(Y(\Phi))_F$.
(b) The triangulated category $\pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F$ is generated by its heart $\pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_{F,w=0}$.

Proof. Let $\mathcal{K}$ be the strict, full, additive sub-category of $DM_{\text{Б},c}(Y(\Phi))_F$ of objects, which are finite direct sums of objects isomorphic to $\pi_{\bar{\sigma},*}\mathbb{1}_{\overline{S_\sigma}}(p)[2p]$, for $\sigma \in \mathfrak{S}$ and $p \in \mathbb{Z}$. On the one hand, according to Corollary 4.9 (b), $\mathcal{K}$ generates the triangulated category $\pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F$. On the other hand, all $\overline{S_\sigma}$ being regular, and $\pi_{\bar{\sigma}}$ proper, $\mathcal{K}$ is contained in the category $CHM(Y(\Phi))_F = DM_{\text{Б},c}(Y(\Phi))_{F,w=0}$ (Theorem 3.1). In particular, by orthogonality 2.1 (3) for the motivic weight structure, $\mathcal{K}$ is *negative*, meaning that

$$\mathrm{Hom}_{\pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F}(M_1, M_2[i]) = \mathrm{Hom}_{DM_{\text{Б},c}(Y(\Phi))_F}(M_1, M_2[i]) = 0$$

for any two objects M_1, M_2 of $\mathcal{K}$, and any integer $i > 0$. Therefore, [B1, Thm. 4.3.2 II 1] can be applied to ensure the existence of a weight structure v on $\pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F$, uniquely characterized by the property of containing $\mathcal{K}$ in its heart. Furthermore [B1, Thm. 4.3.2 II 2], the heart of v is equal to the category $\mathcal{K}'$ of retracts of $\mathcal{K}$ in $\pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F$. In particular, it is contained in the heart $CHM(Y(\Phi))_F$ of the motivic weight structure. The existence of weight filtrations 2.1 (4) for the weight structure v then formally implies that

$$\pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_{F,v\le 0} \subset DM_{\text{Б},c}(Y(\Phi))_{F,w\le 0} \ ,$$

and that

$$\pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_{F,v\ge 0} \subset DM_{\text{Б},c}(Y(\Phi))_{F,w\ge 0} \ .$$

We leave it to the reader to prove from this (cmp. [B1, Lemma 1.3.8]) that in fact

$$\pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_{F,v\le 0} = \pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F \cap DM_{\text{Б},c}(Y(\Phi))_{F,w\le 0}$$

and

$$\pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_{F,v\ge 0} = \pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F \cap DM_{\text{Б},c}(Y(\Phi))_{F,w\ge 0} \ .$$

q.e.d.

Remark 4.11. The proof of Corollary 4.10 follows the standard pattern for the construction of (bounded) weight structures. Its abstract ingredients are as follows: $\mathcal{D} \subset \mathcal{C}$ is a full triangulated sub-category of a triangulated category $\mathcal{C}$ underlying a weight structure w, and $\mathcal{D}$ is generated by a family $\mathcal{F}$ of objects lying in the heart $\mathcal{C}_{w=0}$. Then [B1, Thm. 4.3.2 II] w induces a weight structure on $\mathcal{D}$, whose heart equals the category of retracts of the full, additive sub-category generated by $\mathcal{F}$.

It is worthwhile to spell out Corollary 4.10 when $\pi = \mathrm{id}_{S(\mathfrak{S})}$.

Corollary 4.12. *Assume that the closures $\overline{S_\sigma}$ of all strata S_σ, $\sigma \in \mathfrak{S}$ are regular.*
(a) The motivic weight structure on $DM_{\text{Б},c}(S(\mathfrak{S}))_F$ induces a weight structure on $DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F \subset DM_{\text{Б},c}(S(\mathfrak{S}))_F$.
(b) The triangulated category $DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F$ is generated by the heart of its weight structure, i.e., *by the category $CHMT_{\mathfrak{S}}(S(\mathfrak{S}))_F$ of $\mathfrak{S}$-constructible Chow–Tate motives over $S(\mathfrak{S})$.*

5 Construction of semi-primary categories of Chow motives

This section contains our main structural result. We first need to recall some terminology.

Definition 5.1 (cmp. [Le2, Def. 5.16])**.** Let X be a regular scheme. Define the category $CHM^s(X)_F$ of *smooth Chow motives* over X as the strict, full, dense sub-category of $CHM(X)_F$ generated by the motives of the form

$$f_* \mathbb{1}_S(p)[2p] \ ,$$

for proper, smooth morphisms $f : S \to X$, and integers p.

Remark 5.2. Assume that $j : U \hookrightarrow X$ is the immersion of a dense open subscheme of X. Then [O'S2, Prop. 5.1.1] implies that the functor

$$j^* : CHM^s(X)_F \longrightarrow CHM^s(U)_F$$

induces an isomorphism

$$CHM^s(X)^u_F \xrightarrow{\ \sim\ } CHM^s(U)_{X,F} \ ,$$

where $CHM^s(U)_{X,F} \subset CHM^s(U)_F$ denotes the full subcategory of Chow motives admitting a smooth extension to X. This shows that the theory of intermediate extensions on $CHM^s(U)_{X,F}$ is trivial.

It follows from *duality* [CD, Thm. 15.2.4] that the pseudo-Abelian F-linear symmetric tensor category $CHM^s(X)_F$ is *rigid*. The relation of morphisms in $DM_{\text{Б},c}(\,\bullet\,)_F$ to K-theory [CD, Cor. 14.2.14] shows that morphisms in $CHM^s(X)_F$ are given by the classical relation to Chow groups: for proper, smooth morphisms $f : S \to X$, supposed to be of pure relative dimension d_S, and $g : T \to X$, we have

$$\operatorname{Hom}_{CHM^s(X)_F}\big(f_* \mathbb{1}_S(p)[2p], g_* \mathbb{1}_T(q)[2q]\big) = \operatorname{CH}^{d_S+q-p}(S \times_X T) \otimes_{\mathbb{Z}} F \ .$$

Definition 5.3 ([AK, Déf. 9.1.1], cmp. [Ki, Def. 3.7])**.** Let X be a regular scheme, and $M \in CHM^s(X)_F$ a smooth Chow motive over X.
(a) The motive M is *evenly finite dimensional* if there exists a positive integer m such that $\Lambda^m M = 0$.
(b) The motive M is *oddly finite dimensional* if there exists a positive integer m such that $\operatorname{Sym}^m M = 0$.
(c) The motive M is *finite dimensional* if there exists a direct sum decomposition $M = M^+ \oplus M^-$, where M^+ is evenly finite dimensional and M^- is oddly finite dimensional.

Let us get back to the situation considered at the end of Section 4. The proof of the following uses everything said so far.

Main Theorem 5.4. *Let $\pi : S(\mathfrak{S}) \to Y(\Phi)$ be a proper morphism of good stratifications. Assume that all strata Y_φ of $Y(\Phi)$, $\varphi \in \Phi$, and all closures $\overline{S_\sigma}$ of strata S_σ of $S(\mathfrak{S})$, $\sigma \in \mathfrak{S}$ are regular. Furthermore, assume that for all $\sigma \in \mathfrak{S}$ such that S_σ is a stratum of $\pi^{-1}(Y_\varphi)$, $\varphi \in \Phi$, the following holds: the morphism induced by π on S_σ,*

$$\pi_\sigma : S_\sigma \longrightarrow Y_\varphi$$

factors over a scheme B_σ,

$$\pi_\sigma = \pi'_\sigma \circ \pi''_\sigma : S_\sigma \xrightarrow{\pi''_\sigma} B_\sigma \xrightarrow{\pi'_\sigma} Y_\varphi \ ,$$

and the morphisms π'_σ and π''_σ satisfy the following conditions.

$(1)_\sigma$ *π'_σ is proper and smooth, and the smooth Chow motive*

$$\pi'_{\sigma,*}\mathbb{1}_{B_\sigma} \in CHM^s(Y_\varphi)_F \subset CHM(Y_\varphi)_F$$

is finite dimensional.

$(2)_\sigma$ *The motive*

$$\pi''_{\sigma,*}\mathbb{1}_{S_\sigma} \in DM_{\text{Б},c}(B_\sigma)_F$$

belongs to the category $DMT(B_\sigma)_F$ of Tate motives over B_σ.

Then the heart $\pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_{F,w=0}$ of the motivic weight structure w on $\pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F \subset DM_{\text{Б},c}(Y(\Phi))_F$ (Corollary 4.10) is a Wedderburn category. In particular, it is semi-primary.

Proof. Condition $(2)_\sigma$ implies that the functor $\pi''_{\sigma,*}$ respects the categories of Tate motives:

$$\pi''_{\sigma,*}\, DMT(S_\sigma)_F \subset DMT(B_\sigma)_F \ .$$

It follows that

$$\pi_{\sigma,*}\, DMT(S_\sigma)_F \subset \pi'_{\sigma,*}\, DMT(B_\sigma)_F$$

for all $\sigma \in \mathfrak{S}$. Thus, the category $\pi_* DMT_{\mathfrak{S}}(\pi^{-1}(Y_\varphi))_F$, for fixed $\varphi \in \Phi$, is contained in the strict, full, triangulated sub-category $\mathcal{D}_\varphi$ of $DM_{\mathrm{B},c}(Y_\varphi)_F$ generated by the

$$\pi'_{\sigma,*} \mathbb{1}_{B_\sigma}(p) \ ,$$

for all $p \in \mathbb{Z}$ and $\sigma \in \mathfrak{S}$, such that S_σ is a stratum of $\pi^{-1}(Y_\varphi)$. According to Remark 4.11, the motivic weight structure induces a weight structure on $\mathcal{D}_\varphi$, whose heart $\mathcal{D}_{\varphi,w=0}$ is the pseudo-Abelian completion of the strict, full, additive sub-category of $DM_{\mathrm{B},c}(Y_\varphi)_F$ of objects, which are finite direct sums of objects isomorphic to $\pi'_{\sigma,*} \mathbb{1}_{B_\sigma}(p)[2p]$, for $p \in \mathbb{Z}$ and $\sigma \in \mathfrak{S}$ as above. Condition $(1)_\sigma$ says that $\pi'_{\sigma,*} \mathbb{1}_{B_\sigma}$ is a finite dimensional smooth Chow motive. But then so are all $\pi'_{\sigma,*} \mathbb{1}_{B_\sigma}(p)[2p]$, $p \in \mathbb{Z}$: the canonical action of the symmetric group on $\mathbb{1}_{B_\sigma}(p)[2p] \otimes \mathbb{1}_{B_\sigma}(p)[2p]$ is trivial [CD, first part of Prop. 16.2.10]. Hence any decomposition

$$\pi'_{\sigma,*} \mathbb{1}_{B_\sigma} = M^+ \oplus M^- \ ,$$

where M^+ is evenly finite dimensional, and M^- oddly finite dimensional. will indeed induce a decomposition

$$\pi'_{\sigma,*} \mathbb{1}_{B_\sigma}(p)[2p] = M^+(p)[2p] \oplus M^-(p)[2p] \ ,$$

where $M^+(p)[2p]$ is evenly finite dimensional, and $M^-(p)[2p]$ oddly finite dimensional. By [AK, Prop. 9.1.12] (cmp. [Ki, Prop. 6.9]), all objects of the category $\mathcal{D}_{\varphi,w=0}$ are finite dimensional. The same is therefore true for the objects of the category $\pi_* DMT_{\mathfrak{S}}(\pi^{-1}(Y_\varphi))_{F,w=0}$. By [AK, Thm. 9.2.2], the latter is a Wedderburn category, for all $\varphi \in \Phi$.

Now apply Corollary 4.9 (a) and Theorem 2.9 (b). **q.e.d.**

For future reference, let us extract the following from the proof of Main Theorem 5.4.

Lemma 5.5. *Keep the notation and the hypotheses of Main Theorem 5.4, and fix $\varphi \in \Phi$. Then every object of*

$$\pi_* DMT_{\mathfrak{S}}(\pi^{-1}(Y_\varphi))_{F,w=0}$$

is a direct factor of a finite direct sum of objects isomorphic to $\pi'_{\sigma,*} \mathbb{1}_{B_\sigma}(p)[2p]$, for $p \in \mathbb{Z}$ and $\sigma \in \mathfrak{S}$, such that S_σ is a stratum of $\pi^{-1}(Y_\varphi)$.

Main Theorem 5.4 yields categories of Chow motives, to which the theory of intermediate extensions developed in Section 2 applies (see in particular Summary 2.12). For better legibility, let us fix the hypotheses.

Assumption 5.6. Let $\pi : S(\mathfrak{S}) \to Y(\Phi)$ be a proper morphism of good stratifications. Let Φ_U be an open subset of Φ, meaning that it gives rise to a good stratification $Y(\Phi_U)$, and that $Y(\Phi_U) \hookrightarrow Y(\Phi)$ is an open immersion. Denote by Φ_Z its complement. Write $\mathfrak{S}_U := \pi^{-1}\Phi_U$ and $\mathfrak{S}_Z := \pi^{-1}\Phi_Z$.
(a) For all $\sigma \in \mathfrak{S}$, the closures $\overline{S_\sigma}$ of strata S_σ are regular.

(b) For all $\varphi \in \Phi_Z$, the strata Y_φ are regular.

(c) For all $\varphi \in \Phi_Z$ and $\sigma \in \mathfrak{S}_Z$ such that S_σ is a stratum of $\pi^{-1}(Y_\varphi)$, the morphism $\pi_\sigma : S_\sigma \to Y_\varphi$ can be factorized,

$$\pi_\sigma = \pi'_\sigma \circ \pi''_\sigma : S_\sigma \xrightarrow{\pi''_\sigma} B_\sigma \xrightarrow{\pi'_\sigma} Y_\varphi \ ,$$

such that conditions 5.4 $(1)_\sigma$ and $(2)_\sigma$ are satisfied.

Note that only Assumption 5.6 (a) is needed for the results on gluing (Section 4). In particular (Corollary 4.9), $\pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F$ is obtained by gluing $\pi_* DMT_{\mathfrak{S}_U}(S(\mathfrak{S}_U))_F$ and $\pi_* DMT_{\mathfrak{S}_Z}(S(\mathfrak{S}_Z))_F$.

Corollary 5.7. *Let $\pi : S(\mathfrak{S}) \to Y(\Phi)$ be a proper morphism of good stratifications. Let $\Phi_U \subset \Phi$ be open, and denote by $\mathfrak{S}_U \subset \mathfrak{S}$ the pre-image of Φ_U under π. Suppose that Assumption 5.6 holds. Then the hypotheses of Theorem 2.9 (a) are fulfilled for $\mathcal{C}(X) = \pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F$, $\mathcal{C}(U) = \pi_* DMT_{\mathfrak{S}_U}(S(\mathfrak{S}_U))_F$ and $\mathcal{C}(Z) = \pi_* DMT_{\mathfrak{S}_Z}(S(\mathfrak{S}_Z))_F$. In particular, the functor*

$$j_{!*} : \pi_* DMT_{\mathfrak{S}_U}(S(\mathfrak{S}_U))_{F,w=0} \hookrightarrow \pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))^u_{F,w=0}$$

is defined, and it satisfies the properties listed in Summary 2.12.

Proof. Main Theorem 5.4, applied to $\pi_Z : S(\mathfrak{S}_Z) \to Y(\Phi_Z)$, tells us that $\pi_* DMT_{\mathfrak{S}_Z}(S(\mathfrak{S}_Z))_{F,w=0}$ is semi-primary. **q.e.d.**

In the applications we have in mind, the morphisms π''_σ will turn each S_σ into a torsor under a split torus over B_σ. We leave it to the reader to prove, using the $\mathbb{A}^1$-*homotopy property*, that in this case, condition 5.4 $(2)_\sigma$ holds. Let us now identify concrete situations where condition 5.4 $(1)_\sigma$ is satisfied. The following principles turn out to be very useful.

Proposition 5.8 ([O'S2, pp. 54–55])**.** *Let $g : X' \to X$ be a morphism of regular schemes, $M \in CHM^s(X)_F$ and $M' \in CHM^s(X')_F$.*

(a) If M is finite dimensional (resp. evenly finitely dimensional, resp. oddly finitely dimensional), then so is $g^* M \in CHM^s(X')_F$.

(b) If g is dominant, and $g^* M$ is finite dimensional (resp. evenly finitely dimensional, resp. oddly finitely dimensional), then so is M.

(c) If g is finite, étale, and M' is finite dimensional (resp. evenly finitely dimensional, resp. oddly finitely dimensional), then so is $g_* M' \in CHM^s(X)_F$.

Examples 5.9. Let $f : S \to X$ be a proper, smooth morphism between regular schemes. We give a list of cases where $M := f_* \mathbb{1}_S$ is finite dimensional.

(a) The morphism f is finite and étale (in which case M is an *Artin motive* over X). Indeed, M is then evenly finite dimensional according to Proposition 5.8 (c).

(b) S is an Abelian scheme over X *via* f. Indeed, as observed in [O'S2, lower

half of p. 61], Proposition 5.8 (b) allows to reduce to the case where X is the spectrum of an algebraically closed field. The claim then follows from a classical result of Shermenev [Sh, Theorem]. It generalizes [Kü, Thm. (3.3.1)], which was proved using the Fourier–Mukai transform for Abelian schemes, and assuming that X is quasi-projective and smooth over a field.

(c) More generally (cmp. [Sh, Claim 11 on p. 48]), each fibre of f over a geometric generic point of X is the target of a proper, surjective morphism, whose source is a product of smooth, projective curves. Indeed, apply Proposition 5.8 (b) and [Ki, Prop. 6.9, Rem. 6.6, Cor. 5.11 and Cor. 4.4].

(d) Still more generally, and still thanks to Proposition 5.8 (b), M is finite dimensional as soon as it is isomorphic to one of the earlier cases after a dominant base change. For example, $M = f_* \mathbb{1}_S$ is finite dimensional if S is a torsor under an Abelian scheme over X (for the $fpqc$-topology, say).

The following consequence of Proposition 5.8 (c) will be needed in the sequel.

Corollary 5.10. *Let $\pi : S(\mathfrak{S}) \to Y(\Phi)$ be a morphism of good stratifications. Assume that all strata Y_φ of $Y(\Phi)$, $\varphi \in \Phi$ are regular. Let $a : Y(\Phi) \to W(\Phi)$ be a surjective finite morphism of good stratifications, with the same index set Φ: $a^{-1}(W_\varphi) = Y_\varphi$ for all $\varphi \in \Phi$. Assume that the morphisms induced by a on Y_φ,*

$$a_\varphi : Y_\varphi \longrightarrow W_\varphi \ ,$$

are all finite and étale (hence all W_φ, $\varphi \in \Phi$ are regular). Furthermore, assume that for all $\sigma \in \mathfrak{S}$ such that S_σ is a stratum of $\pi^{-1}(Y_\varphi)$, $\varphi \in \Phi$, a factorization of $\pi_\sigma : S_\sigma \to Y_\varphi$ is given,

$$\pi_\sigma = \pi'_\sigma \circ \pi''_\sigma : S_\sigma \stackrel{\pi''_\sigma}{\longrightarrow} B_\sigma \stackrel{\pi'_\sigma}{\longrightarrow} Y_\varphi \ .$$

Then if all π'_σ satisfy condition 5.4 $(1)_\sigma$, then so do all $a \circ \pi'_\sigma$. In particular, if the morphism π satisfies the hypotheses of Main Theorem 5.4, then so does the morphism $a \circ \pi : S(\mathfrak{S}) \to W(\Phi)$.

6 Compatibility with certain direct and inverse images

In this section, we aim at compatibility statements of type "$a_* \circ j_{!*} = j_{!*} \circ a_*$" and "$b^* \circ j_{!*} = j_{!*} \circ b^*$", for finite morphisms a satisfying certain conditions (Theorem 6.6), and certain smooth morphisms b (Theorem 6.7). Actually, Theorem 6.6 should hold for arbitrary finite a, as is suggested by [BBD, Cor. 4.1.3]: in the context of triangulated categories of sheaves, a_* is exact for the (perverse) t-structure. Similarly remarks apply to Theorem 6.7. Our starting point is the following result.

Proposition 6.1 ([O'S2, p. 55]). *Assume that F is a field of characteristic zero. Let $g : X' \to X$ be a dominant morphism of regular connected schemes. Then the inverse image $g^* : CHM^s(X)_F \to CHM^s(X')_F$ maps the maximal tensor ideal $\mathcal{N}_X$ of $CHM^s(X)_F$ to the maximal tensor ideal $\mathcal{N}_{X'}$ of $CHM^s(X')_F$:*

$$g^*(\mathcal{N}_X) \subset \mathcal{N}_{X'} .$$

A word of explanation is in order: due to the restrictions on F, X and X', the categories $CHM^s(X)_F$ and $CHM^s(X')_F$ are not only F-linear rigid symmetric tensor categories, but they also satisfy the additional hypothesis from [AK, Sect. 7]: F is a field,

$$F = \mathrm{End}_{CHM^s(X)_F}(\mathbb{1}_X) \quad \text{and} \quad F = \mathrm{End}_{CHM^s(X')_F}(\mathbb{1}_{X'})$$

[CD, Cor. 14.2.14]. Therefore, [AK, Prop. 7.1.4 b)] can be applied, ensuring existence and unicity of a *maximal tensor ideal, i.e.*, an ideal $\mathcal{N}_X$, resp. $\mathcal{N}_{X'}$, which is maximal among the tensor ideals unequal to $CHM^s(X)_F$, resp. $CHM^s(X')_F$.

With an appropriate definition of the tensor ideal $\mathcal{N}_\bullet$, Proposition 6.1 admits an obvious generalization to the setting we have considered so far (F a finite product of fields of characteristic zero, schemes which are not necessarily irreducible). Denote by $CHM^s(\bullet)_F^{fd}$ the full sub-category of $CHM^s(\bullet)_F$ formed by finite dimensional objects. By [AK, Thm. 9.1.12, Prop. 9.1.4 b)] (cmp. [Ki, Prop. 6.9, Prop. 5.10]), $CHM^s(\bullet)_F^{fd}$ is a dense tensor sub-category of $CHM^s(\bullet)_F$. It remains rigid according to [AK, Prop. 9.1.4 c)].

Corollary 6.2. *Let $g : X' \to X$ be a morphism of regular schemes, whose restriction to any connected component of X' is dominant over some connected component of X. Then $g^* : CHM^s(X)_F^{fd} \to CHM^s(X')_F^{fd}$ is radicial (see Definition 2.16).*

Proof. According to [AK, Thm. 9.2.2], the ideal $\mathcal{N}_\bullet$ of $CHM^s(\bullet)_F^{fd}$ coincides with the radical. The claim then follows from Proposition 6.1.

q.e.d.

Corollary 6.3. *Let $g : X' \to X$ be a finite, étale morphism of regular schemes. Then both the inverse image $g^* : CHM^s(X)_F^{fd} \to CHM^s(X')_F^{fd}$ and the direct image $g_* : CHM^s(X')_F^{fd} \to CHM^s(X)_F^{fd}$ (Proposition 5.8 (c)) are radicial.*

Proof. For g^*, we have the more general statement of Corollary 6.2.

In order to prove the claim for g_*, note first that it suffices to show it after a surjective base change $f : Y \to X$ with regular source Y: indeed, according to [CD, Thm. 14.3.3], the categories $DM_{\text{Б},c}(\bullet)_F$ are *separated* in the sense of

[CD, Def. 2.1.7]. Therefore, the inverse image f^* is conservative. According to Corollary 6.2, it is radicial on $CHM^s(\bullet)^{fd}_F$. By [AK, Lemme 1.4.7],

$$f^* : CHM^s(X)^{fd}_F \longrightarrow CHM^s(Y)^{fd}_F$$

detects elements in the radical.

Then, taking f to be an appropriate finite, étale morphism, we may assume that X' is a finite disjoint union of copies of X, and g an isomorphism on each component. In that case, our claim follows from [AK, Cor. 1.4.5].

q.e.d.

Let us now show radiciality of a_*, for morphisms a of the type considered at the end of Section 5.

Corollary 6.4. *Let $\pi : S(\mathfrak{S}) \to Y(\Phi)$ be a proper morphism of good stratifications. Assume that all strata Y_φ of $Y(\Phi)$, $\varphi \in \Phi$, and all closures $\overline{S_\sigma}$ of strata S_σ of $S(\mathfrak{S})$, $\sigma \in \mathfrak{S}$ are regular. Furthermore, assume that for all $\sigma \in \mathfrak{S}$ such that S_σ is a stratum of $\pi^{-1}(Y_\varphi)$, the morphism $\pi_\sigma : S_\sigma \to Y_\varphi$ can be factorized,*

$$\pi_\sigma = \pi'_\sigma \circ \pi''_\sigma : S_\sigma \xrightarrow{\pi''_\sigma} B_\sigma \xrightarrow{\pi'_\sigma} Y_\varphi \ ,$$

such that conditions 5.4 $(1)_\sigma$ and $(2)_\sigma$ are satisfied. Let $a : Y(\Phi) \to W(\Phi)$ be a surjective finite morphism of good stratifications, with the same index set Φ. Assume that the morphisms $a_\varphi : Y_\varphi \to W_\varphi$ are all finite and étale. Then the direct image

$$a_* : \pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_{F,w=0} \longrightarrow (a \circ \pi)_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_{F,w=0}$$

is radicial.

By definition, a_* maps $\pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F$ to $(a \circ \pi)_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F$. Since a is finite, hence proper, it is w-exact; therefore, it respects the hearts.

Proof of Corollary 6.4. According to Corollary 4.9 (a), the categories $\pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F$ and $(a \circ \pi)_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F$ are obtained by gluing. Thanks to Proposition 2.17 (b), we are reduced to showing that each

$$a_{\varphi,*} : \pi_* DMT_{\mathfrak{S}}(\pi^{-1}(Y_\varphi))_{F,w=0} \longrightarrow (a \circ \pi)_* DMT_{\mathfrak{S}}(\pi^{-1}(Y_\varphi))_{F,w=0}$$

is radicial. By Lemma 5.5, the source of $a_{\varphi,*}$ is contained in $CHM^s(Y_\varphi)^{fd}_F$, and the target in $CHM^s(W_\varphi)^{fd}_F$. Our claim thus follows from our assumption on a_φ, and from Corollary 6.3. **q.e.d.**

We are ready to state the main results of this section. Again, let us fix the hypotheses.

Assumption 6.5. Let $\pi : S(\mathfrak{S}) \to Y(\Phi)$ and $a : Y(\Phi) \to W(\Phi)$ be proper, resp. surjective finite morphisms of good stratifications. Let Φ_U be an open subset of Φ, and denote by Φ_Z its complement. Write $\mathfrak{S}_U := \pi^{-1}\Phi_U$ and $\mathfrak{S}_Z := \pi^{-1}\Phi_Z$.

(a) For all $\sigma \in \mathfrak{S}$, the closures $\overline{S_\sigma}$ of strata S_σ are regular.
(b) For all $\varphi \in \Phi_Z$, the strata W_φ and Y_φ are regular.
(c) For all $\varphi \in \Phi_Z$, the morphism $a_\varphi : Y_\varphi \to W_\varphi$ is finite and étale.
(d) For all $\varphi \in \Phi_Z$ and $\sigma \in \mathfrak{S}_Z$ such that S_σ is a stratum of $\pi^{-1}(Y_\varphi)$, the morphism $\pi_\sigma : S_\sigma \to Y_\varphi$ can be factorized,

$$\pi_\sigma = \pi'_\sigma \circ \pi''_\sigma : S_\sigma \xrightarrow{\pi''_\sigma} B_\sigma \xrightarrow{\pi'_\sigma} Y_\varphi \, ,$$

such that conditions 5.4 $(1)_\sigma$ and $(2)_\sigma$ are satisfied.

Theorem 6.6. *Let $\pi : S(\mathfrak{S}) \to Y(\Phi)$ and $a : Y(\Phi) \to W(\Phi)$ be proper, resp. surjective finite morphisms of good stratifications. Let Φ_U be an open subset of Φ. Denote by $\mathfrak{S}_U \subset \mathfrak{S}$ the pre-image of Φ_U under π, and by $\mathfrak{S}_Z$ its complement. Suppose that Assumption 6.5 holds. Then the functor*

$$a_{Z,*} : \pi_* DMT_{\mathfrak{S}_Z}(S(\mathfrak{S}_Z))_{F,w=0} \longrightarrow (a \circ \pi)_* DMT_{\mathfrak{S}_Z}(S(\mathfrak{S}_Z))_{F,w=0}$$

is radicial, and the diagram

$$\begin{CD} \pi_* DMT_{\mathfrak{S}_U}(S(\mathfrak{S}_U))_{F,w=0} @>{j_{!*}}>> \pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))^u_{F,w=0} \\ @V{a_{U,*}}VV @VV{a^u_*}V \\ (a \circ \pi)_* DMT_{\mathfrak{S}_U}(S(\mathfrak{S}_U))_{F,w=0} @>{j_{!*}}>> (a \circ \pi)_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))^u_{F,w=0} \end{CD}$$

commutes.

Proof. When applied to $\pi_Z : S(\mathfrak{S}_Z) \to Y(\Phi_Z)$ and $a_Z : Y(\Phi_Z) \to W(\Phi_Z)$, Corollary 6.4 tells us that $a_{Z,*}$ is indeed radicial . Note that according to Main Theorem 5.4 and Corollary 5.10, source and target of $a_{Z,*}$ are semi-primary.
Now apply Proposition 2.17 (a). **q.e.d.**

The proof of the following result is along the same lines; we leave the details to the reader.

Theorem 6.7. *Let $\pi : S(\mathfrak{S}) \to Y(\Phi)$ and $b : Y'(\Phi) \to Y(\Phi)$ be proper, resp. smooth morphisms of good stratifications. Denote by $\pi' : S'(\mathfrak{S}) \to Y'(\Phi)$ the base change of π to $Y'(\Phi)$. Suppose that Assumption 5.6 holds. Then the functor*

$$b^*_Z : \pi_* DMT_{\mathfrak{S}_Z}(S(\mathfrak{S}_Z))_{F,w=0} \longrightarrow \pi'_* DMT_{\mathfrak{S}_Z}(S'(\mathfrak{S}_Z))_{F,w=0}$$

is radicial, and the diagram

$$\begin{CD} \pi_* DMT_{\mathfrak{S}_U}(S(\mathfrak{S}_U))_{F,w=0} @>{j_{!*}}>> \pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))^u_{F,w=0} \\ @V{b^*_U}VV @VV{b^{*,u}}V \\ \pi'_* DMT_{\mathfrak{S}_U}(S'(\mathfrak{S}_U))_{F,w=0} @>{j_{!*}}>> \pi'_* DMT_{\mathfrak{S}}(S'(\mathfrak{S}))^u_{F,w=0} \end{CD}$$

commutes.

Remark 6.8. Theorem 6.7 applies in particular to morphisms b induced by smooth changes $\mathbb{B}' \to \mathbb{B}$ of finite type of our base scheme (see the conventions fixed in our Introduction). Actually, thanks to *continuity* [CD, Prop. 14.3.1], the statement remains valid without the finiteness condition on the base change. This is true for example when $\mathbb{B}' \to \mathbb{B}$ is the inclusion of a generic point of $\mathbb{B}$.

7 Compatibility with the Betti realization

Throughout this section, we fix a generic point $\mathbf{Spec}\, k$ of our base scheme $\mathbb{B}$. We assume k to be of characteristic zero, and to be embeddable into $\mathbb{C}$. We fix such an embedding, and thus get a geometric point

$$\eta : \mathbf{Spec}\, \mathbb{C} \longrightarrow \mathbf{Spec}\, k \hookrightarrow \mathbb{B}$$

of $\mathbb{B}$. For any scheme X in the sense of the conventions fixed in the Introduction, the base change $X \times_{\mathbb{B}} \mathbf{Spec}\, k$ is separated and of finite type over k. We denote by $X(\mathbb{C})$ the set of points of X with values in $\mathbb{C}$ with respect to η. The set $X(\mathbb{C})$ is equipped with the analytic topology.

The *Betti realization* is defined in [Ay2, Déf. 2.1]. It is a family of covariant exact functors

$$R_Y : \mathbf{SH}_{\mathcal{M}}(Y) \longrightarrow D(Y(\mathbb{C}), \mathbb{Z}) ,$$

indexed by quasi-projective k-schemes Y. The source of R_Y is the *stable homotopy category of Y-schemes* [Ay1, Sect. 4.5]. Its target is the (unbounded) derived category of the Abelian category $\mathbf{Shv}(Y(\mathbb{C}))$ of sheaves with values in Abelian groups on the topological space $Y(\mathbb{C})$. The functors R_Y are tensor functors and respect the unit objects [Ay2, Lemme 2.2]. According to [Ay2, Prop. 2.4, Thm. 3.4, Thm. 3.7], they commute with the functors f^*, f_*, $f^!$, $f_!$, provided the latter are applied to constructible objects (note that commutation holds without this restriction for the two functors f^* and $f_!$). In particular, they commute with Tate twists. In [CD, Ex. 17.1.7], it is shown how to obtain from the R_Y a family of exact functors with analogous properties, and which we denote by the same symbols

$$R_Y : DM_{\text{Б},c}(Y) \longrightarrow D^b_c(Y(\mathbb{C}), \mathbb{Q}) ,$$

where the right hand denotes the full triangulated sub-category of $D(Y(\mathbb{C}), \mathbb{Q})$ of classes of bounded complexes with constructible cohomology objects. The construction can be imitated to obtain F-linear versions of the Betti realization. Composing with the base change *via* $\mathbf{Spec}\, k \hookrightarrow \mathbb{B}$, we finally obtain a family of exact tensor functors

$$R_X = R_{\eta,X} : DM_{\text{Б},c}(X)_F \longrightarrow D^b_c(X(\mathbb{C}), F) ,$$

still referred to as the Betti realization, and indexed by schemes X, whose base change $X \times_{\mathbb{B}} \mathbf{Spec}\, k$ is quasi-projective over k. The R_X respect the unit objects. They commute with the functors f^*, f_*, $f^!$, $f_!$ since $\mathbf{Spec}\, k \hookrightarrow \mathbb{B}$ is a projective limit of open immersions (use [CD, Prop. 14.3.1]).

Assumption 7.1. Let $\pi : S(\mathfrak{S}) \to Y(\Phi)$ be a proper morphism of good stratifications of schemes $S(\mathfrak{S})$ and $Y(\Phi)$. Assume that the base change $Y(\Phi) \times_{\mathbb{B}} \mathbf{Spec}\, k$ is quasi-projective over $\mathbf{Spec}\, k$. Let Φ_U be an open subset of Φ, and denote by Φ_Z its complement. Write $\mathfrak{S}_U := \pi^{-1}\Phi_U$ and $\mathfrak{S}_Z := \pi^{-1}\Phi_Z$.
(a) For all $\sigma \in \mathfrak{S}$, the closures $\overline{S_\sigma}$ of strata S_σ are regular.
(b) For all $\varphi \in \Phi_Z$, the strata Y_φ are regular.
(c) For all $\varphi \in \Phi_Z$ and $\sigma \in \mathfrak{S}_Z$ such that S_σ is a stratum of $\pi^{-1}(Y_\varphi)$, the morphism $\pi_\sigma : S_\sigma \to Y_\varphi$ can be factorized,

$$\pi_\sigma = \pi'_\sigma \circ \pi''_\sigma : S_\sigma \xrightarrow{\pi''_\sigma} B_\sigma \xrightarrow{\pi'_\sigma} Y_\varphi \ ,$$

such that condition 5.4 $(2)_\sigma$ is satisfied, such that π'_σ is proper and smooth, and such that the pull-back of the base change

$$\pi'_\sigma \times_{\mathbb{B}} \mathbf{Spec}\, k : B_\sigma \times_{\mathbb{B}} \mathbf{Spec}\, k \longrightarrow Y_\varphi \times_{\mathbb{B}} \mathbf{Spec}\, k$$

to any geometric point of $Y_\varphi \times_{\mathbb{B}} \mathbf{Spec}\, k$ lying over a generic point is isomorphic to a finite disjoint union of Abelian varieties.

Assumption 7.1 is a variant of Assumption 5.6; it is more restrictive in order to respect the hypothesis from [Ay2] on quasi-projectivity, and also to guarantee the validity of the main result of this section, which we state now. For any quasi-projective k-scheme Y, we denote by $\mathbf{Perv}_c(Y(\mathbb{C}), F)$ the heart of the (middle) perverse t-structure [BBD, p. 63] on $D^b_c(Y(\mathbb{C}), F)$, and by

$$H^m : D^b_c(Y(\mathbb{C}), F) \longrightarrow \mathbf{Perv}_c(Y(\mathbb{C}), F) \ , \ m \in \mathbb{Z}$$

the perverse cohomology functors. Denote by $j_{!*} : \mathbf{Perv}_c(Y(\Phi_U)(\mathbb{C}), F) \hookrightarrow \mathbf{Perv}_c(Y(\Phi)(\mathbb{C}), F)$ the intermediate extension of perverse sheaves [BBD, Déf. 1.4.22].

Theorem 7.2. *Let $\pi : S(\mathfrak{S}) \to Y(\Phi)$ be a proper morphism of good stratifications of schemes $S(\mathfrak{S})$ and $Y(\Phi)$. Let Φ_U be an open subset of Φ. Denote by $\mathfrak{S}_U \subset \mathfrak{S}$ the pre-image of Φ_U under π, and by $\mathfrak{S}_Z$ and Φ_Z the respective complements. Suppose that Assumption 7.1 holds.*
(a) For any integer m, the restriction of the composition

$$H^m \circ R_{Y(\Phi)} : DM_{\mathrm{B},c}(Y(\Phi))_F \longrightarrow \mathbf{Perv}_c(Y(\Phi)(\mathbb{C}), F)$$

to $\pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_{F,w=0}$ factors over $\pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))^u_{F,w=0}$.

(b) For any integer m, the diagram

$$\begin{tikzcd}
\pi_* DMT_{\mathfrak{S}_U}(S(\mathfrak{S}_U))_{F,w=0} \arrow[r, hook, "j_{!*}"] \arrow[d, "H^m \circ R_{Y(\Phi_U)}"'] & \pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))^u_{F,w=0} \arrow[d, "H^m \circ R_{Y(\Phi)}"] \\
\mathbf{Perv}_c(Y(\Phi_U)(\mathbb{C}), F) \arrow[r, hook, "j_{!*}"] & \mathbf{Perv}_c(Y(\Phi)(\mathbb{C}), F)
\end{tikzcd}$$

commutes.

The rest of this section is devoted to the proof of Theorem 7.2. As was the case for the proof of the main results of Section 6, it necessitates the study of the statement under gluing. Let us return again to the abstract categorial setting. We fix two triples of F-linear triangulated categories, which are related by gluing: $\mathcal{C}_1(U)$, $\mathcal{C}_1(X)$, $\mathcal{C}_1(Z)$, and $\mathcal{C}_2(U)$, $\mathcal{C}_2(X)$, $\mathcal{C}_2(Z)$ (we use the same symbols j^*, i_* *etc.* for the two sets of gluing functors). The $\mathcal{C}_1(\bullet)$ are assumed pseudo-Abelian, and equipped with a weight structure w, but this time the $\mathcal{C}_2(\bullet)$ are supposed to carry a t-structure t. Both w and t are assumed to be compatible with the gluing. Denote by $\mathcal{C}_2(\bullet)^{t=0}$ the heart of the latter, and by $H^m : \mathcal{C}_2(\bullet) \to \mathcal{C}_2(\bullet)^{t=0}$, $m \in \mathbb{Z}$, the cohomology functors associated to t. Furthermore, F-linear exact functors

$$r_U : \mathcal{C}_1(U) \longrightarrow \mathcal{C}_2(U) \ , \ r_X : \mathcal{C}_1(X) \longrightarrow \mathcal{C}_2(X) \ , \ r_Z : \mathcal{C}_1(Z) \longrightarrow \mathcal{C}_2(Z)$$

are supposed to be given, and to commute with the gluing functors. The following should be seen as an analogue of Proposition 2.17 (b) in the absence of a weight structure on $\mathcal{C}_2(\bullet)$.

Proposition 7.3. *Assume that $\mathcal{C}_1(Z)_{w=0}$ is semi-primary, and that for any object $M \in \mathcal{C}_1(X)_{w=0}$ without non-zero direct factors in $i_*\mathcal{C}_1(Z)_{w=0}$, and any integer m, both*

$$H^0 i^! H^m(r_X M) \quad \textit{and} \quad H^0 i^* H^m(r_X M)$$

are zero. Denote the restriction of $r_\bullet$ to $\mathcal{C}_1(\bullet)_{w=0}$ by $r_{\bullet,w=0}$. If both $r_{Z,w=0}$ and $r_{U,w=0}$ map the radical to the kernel of $\oplus_{m\in\mathbb{Z}} H^m$, then so does $r_{X,w=0}$.

Proof. Let M and N be objects of $\mathcal{C}_1(X)_{w=0}$, and

$$f : M \longrightarrow N$$

a morphism belonging to the radical. According to Theorem 2.9 (a), we may assume that M belongs to the image of $j_{!*}$ (meaning that it does not admit non-zero direct factors in the image of i_*) or to the image of i_*, and likewise for N.

If both M and N are in the image of i_*, then the claim is identical to our assumption on $r_{Z,w=0}$. Now let us treat the case where M has no non-zero direct factors in the image of i_*. Since j^*f is in the radical of $\mathcal{C}_1(U)_{w=0}$,

$$j^* H^m(r_X f) = H^m(r_U(j^* f)) = 0$$

for all $m \in \mathbb{Z}$, by our assumption on $r_{U,w=0}$. It follows that for all m, the morphism $H^m(r_X f)$ in $\mathcal{C}_2(X)^{t=0}$ factors through $i_* H^0 i^* H^m(r_X M) = 0$. The

case where N is without non-zero direct factors in $i_* \mathcal{C}_1(Z)_{w=0}$, is treated dually. **q.e.d.**

Now recall [B1, Prop. 2.1.2 1] that any covariant additive functor $\mathcal{H}$ from a triangulated category $\mathcal{C}$ carrying a weight structure w, to an Abelian category $\mathfrak{A}$ admits a canonical *weight filtration* by sub-functors

$$\ldots \subset W_n \mathcal{H} \subset W_{n+1} \mathcal{H} \subset \ldots \subset \mathcal{H} .$$

According to [B1, Def. 2.1.1] (use the normalization of [B2, Def. 1.3.1] for the signs of the weights), for an object M of $\mathcal{C}$, and $n \in \mathbb{Z}$, the sub-object $W_n \mathcal{H}(M) \subset \mathcal{H}(M)$ is defined as the image of the morphism $\mathcal{H}(\iota_{w \le n})$, for *any* weight filtration

$$M_{w \le n} \stackrel{\iota_{w \le n}}{\longrightarrow} M \longrightarrow M_{w \ge n+1} \longrightarrow M_{w \le n}[1]$$

(with $M_{w \le n} \in \mathcal{C}_{w \le n}$ and $M_{w \ge n+1} \in \mathcal{C}_{w \ge n+1}$). For any $m \in \mathbb{Z}$, one defines

$$\mathcal{H}^m : \mathcal{C} \longrightarrow \mathfrak{A} \; , \; M \longrightarrow \mathcal{H}(X[m]) \; ;$$

according to the usual convention, the weight filtration of $\mathcal{H}^m(M)$ *equals* the weight filtration of $\mathcal{H}(X[m])$, *i.e.*, it differs by *décalage* from the intrinsic weight filtration of the covariant additive functor $\mathcal{H}^m$. The following is a direct consequence of the definitions.

Lemma 7.4. *Let $r : \mathcal{C}_1 \to \mathcal{C}_2$ be an F-linear exact functor between F-linear triangulated categories. Assume that $\mathcal{C}_1$ is equipped with a weight structure w, and that $\mathcal{C}_2$ carries a t-structure t. Let*

$$\alpha \in \operatorname{Hom}_{\mathcal{C}_1}(B, A) \; ,$$

for objects $A \in \mathcal{C}_{1,w \le 0}$ and $B \in \mathcal{C}_{1,w \ge 0}$. Assume the following for all integers m: (i) the morphism $H^m \circ r(\alpha)$ in $\mathcal{C}_2^{t=0}$ is strict with respect to the weight filtrations of $H^m \circ r(B)$ and $H^m \circ r(A)$, (ii) $W_m H^m \circ r(\alpha) = 0$.
Then $r(\alpha)$ is zero on cohomology: $H^m \circ r(\alpha) = 0$, $\forall \, m \in \mathbb{Z}$.

Lemma 7.5. *Assume that $\mathcal{C}_1(Z)_{w=0}$ is semi-primary, and that $r_{Z,w=0}$ maps the radical of $\mathcal{C}_1(Z)_{w=0}$ to the kernel of $\oplus_{m \in \mathbb{Z}} H^m$. Let $M \in \mathcal{C}_1(X)_{w=0}$ be without non-zero direct factors in $i_* \mathcal{C}_1(Z)_{w=0}$. Assume that for all $m \in \mathbb{Z}$, the functor $H^m \circ r_Z$ maps the composition*

$$i^! M \longrightarrow i^* M$$

of the two adjunction morphisms to a morphism which is strict with respect to the weight filtrations. Then the composition

$$i^!(r_X M) \longrightarrow i^*(r_X M)$$

is zero on cohomology.

Proof. Apply Lemma 7.4 to $\mathcal{C}_1 := \mathcal{C}_1(Z)$, $\mathcal{C}_2 := \mathcal{C}_2(Z)$, and $\alpha : i^! M \to i^* M$ equal to the composition of the adjunction morphisms. This is indeed

possible: first, hypothesis 7.4 (i) is satisfied by assumption. Second, note that according to Summary 2.12 (a) (3c), the composition

$$L_0 \longrightarrow i^! M \xrightarrow{\alpha} i^* M \longrightarrow N_0$$

belongs to the radical of $\mathcal{C}_1(Z)_{w=0}$, for any pair of objects L_0, N_0 occurring in weight filtrations of $i^! M$ and $i^* M$, respectively. Our assumption on $r_{Z,w=0}$ thus ensures the validity of hypothesis 7.4 (ii).

Lemma 7.4 then tells us that the morphism $r_Z(\alpha)$ is zero on cohomology. But since the functors $r_{\bullet}$ commute with the gluing, the morphism $r_Z(\alpha)$ equals the composition

$$i^!(r_X M) \longrightarrow i^*(r_X M)$$

of the adjunction morphisms. **q.e.d.**

In the context of the Betti realization

$$R_{\bullet} : DM_{\mathrm{B},c}(\bullet)_F \longrightarrow D^b_c(\bullet(\mathbb{C}), F) ,$$

we now have to check the hypotheses of Proposition 7.3 and Lemma 7.5. We shall need the following partial compatibility result.

Proposition 7.6. *Let $f : S \to X$ be a proper morphism of schemes. Assume that the base change $X \times_{\mathbb{B}} \mathbf{Spec}\, k$ is quasi-projective. Then for all integers m, the weight filtration step W_{m-1} of [B1, Def. 2.1.1] on the perverse sheaf*

$$H^m \circ R_X(f_* \mathbb{1}_S)$$

on $X(\mathbb{C})$ coincides with the weight filtration step W_{m-1} underlying the algebraic mixed Hodge module

$$H^m(f_* \mathbb{Q}_{S_{\mathbb{C}}}) .$$

Here, we make use of the forgetful functor

$$D^b\big(\mathbf{MHM}_{\mathbb{Q}}(\bullet \times_{\mathbb{B}} \mathbf{Spec}\, \mathbb{C})\big) \longrightarrow D^b_c(\bullet(\mathbb{C}), F)$$

from the bounded derived category $D^b(\mathbf{MHM}_{\mathbb{Q}}(\bullet \times_{\mathbb{B}} \mathbf{Spec}\, \mathbb{C}))$ of algebraic mixed Hodge modules on $\bullet \times_{\mathbb{B}} \mathbf{Spec}\, \mathbb{C}$ [Sa2, Sect. 4.2]. It is t-exact with respect to the obvious t-structure on $D^b(\mathbf{MHM}_{\mathbb{Q}}(\bullet \times_{\mathbb{B}} \mathbf{Spec}\, \mathbb{C}))$ and the perverse t-structure on $D^b_c(\bullet(\mathbb{C}), F)$, and it commutes with the Grothendieck operations f^*, f_*, $f^!$, $f_!$ on both sides [Sa2, Thm. 4.3, Sect. 4.4].

Proof of Proposition 7.6. Without loss of generality, we may assume that $\mathbb{B} = \mathbf{Spec}\, k$, and that $k = \mathbb{C}$.

Suppose first that S is regular. Then $f_* \mathbb{1}_S$ is a Chow motive (Theorem 3.1 (1), (2)), hence the weight filtration on $H^m \circ R_X(f_* \mathbb{1}_S)$ is concentrated in weight m. But the same is then true for the Hodge theoretical weight filtration [Sa1, Thm. 5.3.1].

In general case, let $s : \tilde{S} \to S$ be a surjective, proper morphism with regular source. Consider the adjunction morphism

$$adj : \mathbb{1}_S \longrightarrow s_* \mathbb{1}_{\tilde{S}} \ .$$

Choose $C \in DM_{\mathrm{B},c}(S)_F$ fitting into an exact triangle

$$C \longrightarrow \mathbb{1}_S \stackrel{adj}{\longrightarrow} s_* \mathbb{1}_{\tilde{S}} \longrightarrow C[1] \ .$$

Our claim follows from the next Lemma. **q.e.d.**

Lemma 7.7. *(a) The motive C lies in $DM_{\mathrm{B},c}(S)_{F,w\le -1}$. The exact triangle*

$$f_* C \longrightarrow f_* \mathbb{1}_S \stackrel{f_* adj}{\longrightarrow} (fs)_* \mathbb{1}_{\tilde{S}} \longrightarrow f_* C[1]$$

is a weight filtration of $f_* \mathbb{1}_S$.
(b) For all integers m, the Hodge theoretical weight filtration step W_{m-1} on the perverse sheaf $H^m \circ R_X(f_* \mathbb{1}_S)$ coincides with the kernel of the morphism induced by $f_* adj$,

$$H^m \circ R_X(f_* \mathbb{1}_S) \longrightarrow H^m \circ R_X((fs)_* \mathbb{1}_{\tilde{S}}) \ .$$

Proof. The second statement of (a) follows from the first, since f is proper (Theorem 3.1 (2)).

For the remaining claims, let us start by assuming S to be regular. In that case, we imitate the proof of [Ad, Prop. 4.2.6.1] in order to show that the morphism *adj* is split monomorphic. This, together with stability of purity under direct images of proper morphisms [Sa1, Thm. 5.3.1] establishes both (a) and (b).

In the general case, we use induction on the dimension d of S. When $d = 0$, then S is regular. Thus assume $d > 0$, and choose a dense, regular open subscheme $j : U \hookrightarrow S$. Denote by $i : Z \hookrightarrow S$ the complementary closed immersion (Z carries the reduced scheme structure, say).

In order to prove claim (a), consider the localization triangles for $\mathbb{1}_S$ and for $s_* \mathbb{1}_{\tilde{S}}$.

$$\begin{array}{ccccccc} j_! \mathbb{1}_U & \longrightarrow & \mathbb{1}_S & \longrightarrow & i_* \mathbb{1}_Z & \xrightarrow{[1]} & \\ {\scriptstyle j_! adj}\downarrow & & {\scriptstyle j_! adj}\downarrow & & {\scriptstyle i_* adj}\downarrow & & \\ j_! s_* \mathbb{1}_{\tilde{U}} & \longrightarrow & s_* \mathbb{1}_{\tilde{S}} & \longrightarrow & i_* s_* \mathbb{1}_{\tilde{Z}} & \xrightarrow{[1]} & \end{array}$$

Here, we denote by $\tilde{U}$ and $\tilde{Z}$ the pre-images under s of U and Z, respectively. Complete this morphism of exact triangles to a diagram with exact rows and

columns.

$$
\begin{tikzcd}
j_! C_U \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & i_* C_Z \arrow[r, "{[1]}"] \arrow[d] & {} \\
j_! \mathbb{1}_U \arrow[r] \arrow[d, "j_! adj"'] & \mathbb{1}_S \arrow[r] \arrow[d, "j_! adj"'] & i_* \mathbb{1}_Z \arrow[r, "{[1]}"] \arrow[d, "i_* adj"'] & {} \\
j_! s_* \mathbb{1}_{\tilde{U}} \arrow[r] \arrow[d, "{[1]}"'] & s_* \mathbb{1}_{\tilde{S}} \arrow[r] \arrow[d, "{[1]}"'] & i_* s_* \mathbb{1}_{\tilde{Z}} \arrow[r, "{[1]}"] \arrow[d, "{[1]}"'] & {} \\
{} & {} & {} &
\end{tikzcd}
$$

According to the regular case, C_U is of weights ≤ -1, hence so is $j_! C_U$ (Theorem 3.1 (2)). In order to limit the weights occurring in C_Z, hence in $i_* C_Z$, let $z : \tilde{\tilde{Z}} \to \tilde{Z}$ be surjective and proper, with regular source. The composition $sz : \tilde{\tilde{Z}} \to Z$ is also surjective and proper. We complete the diagram

$$
\begin{tikzcd}
\mathbb{1}_Z \arrow[r, "adj"] \arrow[d, "adj"'] & (sz)_* \mathbb{1}_{\tilde{\tilde{Z}}} \arrow[d, equal] \\
s_* \mathbb{1}_{\tilde{Z}} \arrow[r, "s_* adj"] & (sz)_* \mathbb{1}_{\tilde{\tilde{Z}}}
\end{tikzcd}
$$

to a diagram of exact triangles, and apply the induction hypothesis to both $adj : \mathbb{1}_Z \to (sz)_* \mathbb{1}_{\tilde{\tilde{Z}}}$ and $adj : \mathbb{1}_{\tilde{Z}} \to z_* \mathbb{1}_{\tilde{\tilde{Z}}}$. Thus, we see that indeed $C_Z \in DM_{\mathbb{B},c}(Z)_{F, w \leq -1}$.

In order to prove (b), note that the images under R_X of the above diagrams are isomorphic to the images under forgetful functor of the Hodge theoretical analogues of the diagrams. We may thus repeat the reasoning, observing that the analogues of the inequalities for weights from Theorem 3.1 (2) hold in the Hodge theoretical context thanks to [Sa2, (4.5.2)]. **q.e.d.**

Remark 7.8. (a) Lemma 7.7 (b) (is most probably known to the experts and) generalizes the first statement of [D2, Prop. (8.2.5)], which concerns the case $X = \mathbf{Spec}\, \mathbb{C}$.
(b) Using simplicial resolutions as in [D2, Sect. 8.1], it can be shown that the whole weight filtration on

$$
H^m \circ R_X(f_* \mathbb{1}_S)
$$

coincides with the Hodge theoretical weight filtration; this holds more generally for separated morphisms f. This result should be compared to [LW, Prop. 2.2.2], which concerns the case when $X = \mathbb{B} = \mathbf{Spec}\, k$ and f is smooth. In the sequel, we shall only use Proposition 7.6 as stated.

Corollary 7.9. *Let X be a scheme, whose base change $X \times_{\mathbb{B}} \mathbf{Spec}\, k$ is quasi-projective over k. Let $i : Z \hookrightarrow X$ be the immersion of a closed*

subscheme of X. Let $M \in CHM(X)_F$. Then for all $m \in \mathbb{Z}$, the functor $H^m \circ R_Z$ maps the composition

$$i^! M \longrightarrow i^* M$$

of the two adjunction morphisms to a morphism which is strict with respect to the weight filtrations.

Proof. Our claim is clearly stable under passage to direct factors. Given the description of objects of $CHM(X)_F$ from Theorem 3.1 (3), it therefore suffices check the claim for motives M of the form

$$f_* \mathbb{1}_S(p)[2p] \ ,$$

for a projective morphism $f : S \to X$ with regular source S, and an integer p. The claim is invariant under twists and shifts, meaning that we may assume $p = 0$. Furthermore, the base change $S \times_{\mathbb{B}} \mathbf{Spec}\, k$ is quasi-projective over k. The functors $R_\bullet$ commute with f_*, $i^!$ and i^*; therefore, R_Z maps the morphism $i^! M \to i^* M$ to the composition

$$i^! f_*(R_S \mathbb{1}_S) \longrightarrow i^* f_*(R_S \mathbb{1}_S)$$

of the adjunction maps in $D^b_c(\,\bullet\,(\mathbb{C}), F)$. But this composition lies in the image of the forgetful functor

$$D^b(\mathbf{MHM}_{\mathbb{Q}}(Z \times_{\mathbb{B}} \mathbf{Spec}\, \mathbb{C})) \longrightarrow D^b_c(Z(\mathbb{C}), F) \ .$$

Altogether, this establishes that for all $m \in \mathbb{Z}$, the functor $H^m \circ R_Z$ maps the composition

$$i^! f_* \mathbb{1}_S \longrightarrow i^* f_* \mathbb{1}_S$$

of the two adjunction morphisms to a morphism α_m which is strict with respect to the weight filtrations of Hodge theory.

It remains to establish the analogous statement for the weight filtrations of [B1, Def. 2.1.1].

Observe first (Theorem 3.1 (2), [Sa2, (4.5.2)]) that the perverse sheaf $H^m \circ R_Z(i^! f_* \mathbb{1}_S)$ is of weights $\geq m$, whatever of the two notions of weight is chosen. Likewise, $H^m \circ R_Z(i^* f_* \mathbb{1}_S)$ is of weights $\leq m$. Strictness of

$$\alpha_m : H^m \circ R_Z(i^! f_* \mathbb{1}_S) \longrightarrow H^m \circ R_Z(i^* f_* \mathbb{1}_S)$$

is thus equivalent to the following statement: the image of α_m coincides with the image of the restriction of α_m to $W_m H^m \circ R_Z(i^! f_* \mathbb{1}_S)$. This statement being already established for the Hodge theoretical weight filtration, we need to show that the filtration steps W_m coincide on $H^m \circ R_Z(i^! f_* \mathbb{1}_S)$.

Second, apply duality for motives [CD, Thm. 15.2.4 (d)] and for Hodge modules [Sa2, (4.2.3)]. It shows that the desired claim is equivalent to the following: the filtration steps W_{m-1} coincide on $H^m \circ R_Z(i^* f_* \mathbb{1}_S)$ (recall that f is projective, hence proper). By proper base change [CD, Thm. 2.4.50 (4)],

$$i^* f_* \mathbb{1}_S = f'_* \mathbb{1}_{S_Z} \ ,$$

where $f' : S_Z \to Z$ is the base change of f *via* i. Now apply Proposition 7.6.
q.e.d.

The following is a direct consequence of the Decomposition Theorem [BBD, Thm. 6.2.5].

Theorem 7.10. *Let X be a scheme, whose base change $X \times_{\mathbb{B}} \mathbf{Spec}\, k$ is quasi-projective over k. Then the essential image of the restriction of*

$$R_X : DM_{\mathrm{B},c}(X)_F \longrightarrow D^b_c(X(\mathbb{C}), F)$$

to $CHM(X)_F$ is contained in the full sub-category of $D^b_c(X(\mathbb{C}), F)$ of semi-simple *objects: for all $M \in CHM(X)_F$, the following holds.*
(a) The object $R_X M$ is split, i.e., *there exists an isomorphism*

$$\beta : R_X M \xrightarrow{\sim} \bigoplus_{m \in \mathbb{Z}} H^m(R_X M)[-m]$$

in $D^b_c(X(\mathbb{C}), F)$ such that $H^m \beta = \mathrm{id}_{H^m(R_X M)}$, for all $m \in \mathbb{Z}$.
(b) For all $m \in \mathbb{Z}$, the perverse sheaf $H^m(R_X M)$ is a direct sum of simple perverse sheaves.

Proof. An object K of $D^b_c(X(\mathbb{C}), F)$ is split if and only if for all integers m, the boundary morphism δ in the exact truncation triangle

$$\tau_{\le m-1} K \longrightarrow \tau_{\le m} K \longrightarrow H^m(K)[-m] \xrightarrow{\delta} (\tau_{\le m-1} K)[1]$$

is trivial. This shows that the property of being semi-simple is stable under passage to direct factors. As in the proof of Corollary 7.9, we may therefore assume M to equal $f_* \mathbb{1}_S$, for a projective morphism $f : S \to X$ with regular source S. The claim therefore follows from the commutation of $R_\bullet$ with f_*, and from the Decomposition Theorem [BBD, Thm. 6.2.5]. **q.e.d.**

Let us come back to the morphism $\pi : S(\mathfrak{S}) \to Y(\Phi)$. The preceding results allow us to establish the key technical ingredient of the proof of Theorem 7.2, as far as gluing is concerned.

Corollary 7.11. *Let $\pi : S(\mathfrak{S}) \to Y(\Phi)$ be a proper morphism of good stratifications of schemes $S(\mathfrak{S})$ and $Y(\Phi)$. Assume that the base change $Y(\Phi) \times_{\mathbb{B}} \mathbf{Spec}\, k$ is quasi-projective. Let Φ_U be an open subset of Φ. Denote by $\mathfrak{S}_U \subset \mathfrak{S}$ the pre-image of Φ_U under π, and by $\mathfrak{S}_Z$ and Φ_Z the respective complements. Denote by $i : Y(\Phi_Z) \hookrightarrow Y(\Phi)$ the closed immersion. Suppose the validity of Assumption 5.6. Suppose also that the restriction of the functor*

$$R_{Y(\Phi_Z)} : DM_{\mathrm{B},c}(Y(\Phi_Z))_F \longrightarrow D^b_c(Y(\Phi_Z)(\mathbb{C}), F)$$

to the sub-category $\pi_* DMT_{\mathfrak{S}_Z}(S(\mathfrak{S}_Z))_{F,w=0}$ of $DM_{\mathrm{B},c}(Y(\Phi_Z))_F$ maps the radical of $\pi_* DMT_{\mathfrak{S}_Z}(S(\mathfrak{S}_Z))_{F,w=0}$ to the kernel of all perverse cohomology functors

$$H^m : D^b_c(Y(\Phi_Z)(\mathbb{C}), F) \longrightarrow \mathbf{Perv}_c(Y(\Phi_Z)(\mathbb{C}), F)\ ,\ m \in \mathbb{Z}\ .$$

(a) Let $M \in \pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_{F,w=0}$ without non-zero factors in the image of i_*. Then for any integer m, both

$$H^0 i^! H^m(R_{Y(\Phi)} M) \quad and \quad H^0 i^* H^m(R_{Y(\Phi)} M)$$

are zero.

(b) If both

$$R_{Y(\Phi_Z),w=0} : \pi_* DMT_{\mathfrak{S}_Z}(S(\mathfrak{S}_Z))_{F,w=0} \longrightarrow D^b_c(Y(\Phi_Z)(\mathbb{C}), F)$$

and

$$R_{Y(\Phi_U),w=0} : \pi_* DMT_{\mathfrak{S}_U}(S(\mathfrak{S}_U))_{F,w=0} \longrightarrow D^b_c(Y(\Phi_U)(\mathbb{C}), F)$$

map the radical to the kernel of all perverse cohomology functors H^m, then so does

$$R_{Y(\Phi),w=0} : \pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_{F,w=0} \longrightarrow D^b_c(Y(\Phi)(\mathbb{C}), F) \ .$$

Proof. Denote by $j : Y(\Phi_U) \hookrightarrow Y(\Phi)$ the complement of i. Theorem 7.10 and the explicit description of simple perverse sheaves from [BBD, Prop. 1.4.26] show that there is an isomorphism

$$R_{Y(\Phi)} M \cong \bigoplus_{m \in \mathbb{Z}} \big(j_{!*} M_U^m \oplus i_* M_Z^m \big)[-m] \ ,$$

for suitable perverse sheaves M_U^m on $Y(\Phi_U)(\mathbb{C})$ and M_Z^m on $Y(\Phi_Z)(\mathbb{C})$, respectively. Thus, for all integers m,

$$H^m(R_{Y(\Phi)} M) \cong j_{!*} M_U^m \oplus i_* M_Z^m \ ,$$

hence

$$H^0 i^! H^m(R_{Y(\Phi)} M) \cong M_Z^m \cong H^0 i^* H^m(R_{Y(\Phi)} M)$$

[BBD, Cor. 1.4.25], and the latter isomorphisms identify the composition $H^0 i^! H^m(R_{Y(\Phi)} M) \to H^0 i^* H^m(R_{Y(\Phi)} M)$ with the identity on M_Z^m. Thanks to Assumption 5.6, Main Theorem 5.4 can be applied; therefore, the category $\pi_* DMT_{\mathfrak{S}_Z}(S(\mathfrak{S}_Z))_{F,w=0}$ is semi-primary. This, together with Corollary 7.9, shows that the assumptions of Lemma 7.5 are met. Thus, the isomorphism

$$H^0 i^! H^m(R_{Y(\Phi)} M) \xrightarrow{\ \sim\ } H^0 i^* H^m(R_{Y(\Phi)} M)$$

is in fact zero, which shows claim (a). The assumptions of Proposition 7.3 are all satisfied. Therefore, claim (b) holds. **q.e.d.**

The main point that remains to be addressed, concerns the hypothesis "$R_{Y(\Phi_\bullet),w=0}(\mathrm{rad}) \subset \ker H^m$". The following is a reformulation of the main result from [Li].

Theorem 7.12. *Let $p_1 : B_1 \to X$ and $p_2 : B_2 \to X$ be proper, smooth morphisms of schemes. Assume that the base change $X \times_{\mathbb{B}} \mathbf{Spec}\, k$ is quasi-projective, and that the pull-back of*

$$p_r \times_{\mathbb{B}} \mathbf{Spec}\, k : B_r \times_{\mathbb{B}} \mathbf{Spec}\, k \longrightarrow X \times_{\mathbb{B}} \mathbf{Spec}\, k \ , \ r = 1, 2 \ ,$$

to any geometric point of $X \times_{\mathbb{B}} \mathbf{Spec}\, k$ lying over a generic point is isomorphic to a finite disjoint union of Abelian varieties. Then the restriction of the functor

$$R_X : DM_{\mathrm{B},c}(X)_F \longrightarrow D^b_c(X(\mathbb{C}), F)$$

to $CHM(X)_F$ maps the radical

$$\mathrm{rad}_{CHM(X)_F}\bigl(p_{1,*}\mathbb{1}_{B_1}, p_{2,*}\mathbb{1}_{B_2}\bigr) \subset \mathrm{Hom}_{CHM(X)_F}\bigl(p_{1,*}\mathbb{1}_{B_1}, p_{2,*}\mathbb{1}_{B_2}\bigr)$$

to the kernel of all perverse cohomology functors

$$H^m : D^b_c(X(\mathbb{C}), F) \longrightarrow \mathbf{Perv}_c(X(\mathbb{C}), F)\ ,\ m \in \mathbb{Z}\ .$$

Proof. Without loss of generality, we may assume that $\mathbb{B} = \mathbf{Spec}\, k$, and that $k = \mathbb{C}$. Given that the p_r are proper and smooth, the perverse cohomology objects of the $R_X(p_{r,*}\mathbb{1}_{B_r}) = p_{r,*}\mathbb{Q}_{B_{r,\mathbb{C}}}$ are local systems up to a shift. A morphism of local systems on a connected topological space is trivial as soon as it is generically trivial. Therefore, we may assume that $X = \mathbf{Spec}\, \mathbb{C}$, and that the B_r are Abelian varieties, $r = 1, 2$. In particular [Kü, Thm. (3.3.1)], the smooth Chow motives $p_{r,*}\mathbb{1}_{B_r}$ are finite dimensional.

The radical

$$\mathrm{rad}_{CHM(X)_F}\bigl(p_{1,*}\mathbb{1}_{B_1}, p_{2,*}\mathbb{1}_{B_2}\bigr)$$

then coincides with the maximal tensor ideal

$$\mathcal{N}_X\bigl(p_{1,*}\mathbb{1}_{B_1}, p_{2,*}\mathbb{1}_{B_2}\bigr)$$

[AK, Thm. 9.2.2]. It thus consists of the classes of numerically trivial cycles on $B_1 \times_X B_2$ [AK, Ex. 7.1.2]. According to [Li, Thm. 4], numerical and homological equivalence coincide on Abelian varieties over $\mathbb{C}$. In other words,

$$R_X\bigl(\mathrm{rad}_{CHM(X)_F}\bigl(p_{1,*}\mathbb{1}_{B_1}, p_{2,*}\mathbb{1}_{B_2}\bigr)\bigr)$$

consists of morphisms which are zero on cohomology. **q.e.d.**

Corollary 7.13. *Let $\pi : S(\mathfrak{S}) \to Y(\Phi)$ be a proper morphism of good stratifications $S(\mathfrak{S})$ and $Y(\Phi)$. Assume that the base change $Y(\Phi) \times_{\mathbb{B}} \mathbf{Spec}\, k$ is quasi-projective, and that all strata Y_φ of $Y(\Phi)$, $\varphi \in \Phi$, and all closures $\overline{S_\sigma}$ of strata S_σ of $S(\mathfrak{S})$, $\sigma \in \mathfrak{S}$ are regular. Furthermore, assume that for all $\sigma \in \mathfrak{S}$ such that S_σ is a stratum of $\pi^{-1}(Y_\varphi)$, the morphism $\pi_\sigma : S_\sigma \to Y_\varphi$ can be factorized,*

$$\pi_\sigma = \pi'_\sigma \circ \pi''_\sigma : S_\sigma \xrightarrow{\pi''_\sigma} B_\sigma \xrightarrow{\pi'_\sigma} Y_\varphi\ ,$$

such that conditions 5.4 $(1)_\sigma$ and $(2)_\sigma$ are satisfied, and such that the pull-back of the base change

$$\pi'_\sigma \times_{\mathbb{B}} \mathbf{Spec}\, k : B_\sigma \times_{\mathbb{B}} \mathbf{Spec}\, k \longrightarrow Y_\varphi \times_{\mathbb{B}} \mathbf{Spec}\, k$$

to any geometric point of $Y_\varphi \times_{\mathbb{B}} \mathbf{Spec}\, k$ lying over a generic point is isomorphic to a finite disjoint union of Abelian varieties. Then the functor

$$R_{Y(\Phi),w=0} : \pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_{F,w=0} \longrightarrow D^b_c(Y(\Phi)(\mathbb{C}), F)$$

maps the radical of $\pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_{F,w=0}$ to the kernel of all perverse cohomology functors

$$H^m : D^b_c(Y(\Phi)(\mathbb{C}), F) \longrightarrow \mathbf{Perv}_c(Y(\Phi)(\mathbb{C}), F) \ , \ m \in \mathbb{Z} \ .$$

By definition, a proper morphism π is *of Abelian type* if it satisfies the hypotheses of Corollary 7.13. A *Chow motive of Abelian type over $Y(\Phi)$* is a direct factor of an object of $\pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_{F,w=0}$, for a proper morphism $\pi : S(\mathfrak{S}) \to Y(\Phi)$ of Abelian type.

Proof of Corollary 7.13. According to Corollary 4.9 (a), the category $\pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F$ is obtained by gluing. Thanks to Corollary 7.11 (b), we are reduced to showing that each

$$R_{Y_\varphi,w=0} : \pi_* DMT_{\mathfrak{S}}(\pi^{-1}(Y_\varphi))_{F,w=0} \longrightarrow D^b_c(Y_\varphi(\mathbb{C}), F)$$

maps the radical of $\pi_* DMT_{\mathfrak{S}}(\pi^{-1}(Y_\varphi))_{F,w=0}$ to the kernel of all perverse cohomology functors H^m. By Lemma 5.5, all objects of the source of $R_{Y_\varphi,w=0}$ are direct factors of a finite direct sum of objects isomorphic to $\pi'_{\sigma,*}\mathbb{1}_{B_\sigma}(p)[2p]$, for $p \in \mathbb{Z}$ and $\sigma \in \mathfrak{S}$, such that S_σ is a stratum of $\pi^{-1}(Y_\varphi)$. Our claim thus follows from our assumption on π'_σ, the commutation of $R_{Y_\varphi,w=0}$ with Tate twists, and from Theorem 7.12. **q.e.d.**

Proof of Theorem 7.2. When applied to $\pi_Z : S(\mathfrak{S}_Z) \to Y(\Phi_Z)$, Corollary 7.13 tells us that the restriction of the functor

$$R_{Y(\Phi_Z)} : DM_{\text{Б},c}(Y(\Phi_Z))_F \longrightarrow D^b_c(Y(\Phi_Z)(\mathbb{C}), F)$$

to the sub-category $\pi_* DMT_{\mathfrak{S}_Z}(S(\mathfrak{S}_Z))_{F,w=0}$ of $DM_{\text{Б},c}(Y(\Phi_Z))_F$ maps the radical of $\pi_* DMT_{\mathfrak{S}_Z}(S(\mathfrak{S}_Z))_{F,w=0}$ to the kernel of all perverse cohomology functors

$$H^m : D^b_c(Y(\Phi_Z)(\mathbb{C}), F) \longrightarrow \mathbf{Perv}_c(Y(\Phi_Z)(\mathbb{C}), F) \ .$$

(a): Recall (Definition 1.6) that the category $\pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))^u_{F,w=0}$ is the quotient $\pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_{F,w=0}/\mathfrak{g}$, where $\mathfrak{g}$ is the two-sided ideal generated by

$$\mathrm{Hom}_{\pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_{F,w=0}}(M, i_*N) \quad \text{and} \quad \mathrm{Hom}_{\pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_{F,w=0}}(i_*N, M) \ ,$$

for all objects (M,N) of $\pi_* DMT_{\mathfrak{S}}(S(\mathfrak{S}))_{F,w=0} \times \pi_* DMT_{\mathfrak{S}_Z}(S(\mathfrak{S}_Z))_{F,w=0}$, such that M admits no non-zero direct factor in the image of i_*. In order to prove the claim, it thus suffices to show that for all morphisms $f : M \to i_*N$ and $g : i_*N \to M$, for M and N as above, the morphisms $R_{Y(\Phi)}f$ and $R_{Y(\Phi)}g$ are zero on cohomology. But this follows from Theorem 7.10 (a) and Corollary 7.11 (a).

(b): Let $M_U \in \pi_* DMT_{\mathfrak{S}_U}(S(\mathfrak{S}_U))_{F,w=0}$. Then $j_{!*}M_U$ is an extension of M_U without non-zero direct factors in the image of i_*. According to Corollary 7.11 (a), for any integer m, the perverse sheaves

$$H^0 i^! H^m(R_{Y(\Phi)} j_{!*} M_U) \quad \text{and} \quad H^0 i^* H^m(R_{Y(\Phi)} j_{!*} M_U)$$

are zero. Therefore [BBD, Cor. 1.4.24],

$$H^m(R_{Y(\Phi)} j_{!*} M_U) = j_{!*} j^* H^m(R_{Y(\Phi)} j_{!*} M_U) = j_{!*} H^m(R_{Y(\Phi_U)} M_U) \ .$$

This shows commutation of the diagram on the level of objects. Now let f be a morphism in $\pi_* DMT_{\mathfrak{S}_U}(S(\mathfrak{S}_U))_{F,w=0}$. In order to show the relation

$$(H^m \circ R_{Y(\Phi)})(j_{!*} f) = j_{!*} \circ H^m(R_{Y(\Phi_U)} f) \ ,$$

in $\mathbf{Perv}_c(Y(\Phi)(\mathbb{C}), F)$, observe that the relation holds trivially after restriction to $Y(\Phi_U)(\mathbb{C})$. Thus, both $(H^m \circ R_{Y(\Phi)})(j_{!*} f)$ and $j_{!*} \circ H^m(R_{Y(\Phi_U)} f)$ extend the same morphism. But their source and target are intermediate extensions; therefore, morphisms between their restrictions extend uniquely.

q.e.d.

Remark 7.14. (a) A more conceptual, and much less complex proof of Theorem 7.2 could be given if the Hodge theoretical realization

$$R_{\mathbf{H},X} : DM_{\mathrm{B},c}(X)_F \longrightarrow D^b\big(\mathbf{MHM}_{\mathbb{Q}}(X \times_{\mathbb{B}} \mathbf{Spec}\,\mathbb{C}) \otimes_{\mathbb{Q}} F\big)$$

were known to exist, and to yield the Betti realization R_X after composition with the forgetful functor. Indeed, the target of $R_{\mathbf{H},X}$ carries a weight structure (Example 2.3 (b)). Proposition 2.17 would then replace everything leading up to our main gluing result Corollary 7.11 (Proposition 7.3 – Theorem 7.10). Indeed, given Corollary 7.13, the functor $R_{\mathbf{H},X}$ can be expected to be radicial on Chow motives, given that the radical of its target consists precisely of the morphisms, which are zero on cohomology. *Vice versa*, given the interpretation of [Li, Thm. 4] in the proof of Theorem 7.12, to conjecture $R_{\mathbf{H},X}$ to be radicial on the heart of the motivic weight structure seems to be the correct generalization of the conjecture "numerical equivalence equals (Betti) homological equivalence" to Chow motives over arbitrary bases.

(b) The Hodge theoretical realization $R_{\mathbf{H},X}$ left its "shadow" in the present proof of Theorem 7.2, *via* (a) strictness of the Hodge theoretical weight filtration on perverse sheaves of geometric origin (proof of Corollary 7.9), (b) the Decomposition Theorem (proof of Corollary 7.11), which should be considered as a Hodge theoretical phenomenon [Sa2, (4.5.4)].

(c) It is reasonable to expect the analogue of Theorem 7.2 to hold for the ℓ-adic realization [Ay3, Déf. 9.6]. Indeed, when all strata Y_φ are of characteristic zero (meaning that there are no non-zero integers vanishing on Y_φ), one may expect to repeat the above proof, observing that its essential ingredients: strictness of the weight filtration, the Decomposition Theorem, and [Li, Thm. 4], continue to hold for ℓ-adic perverse sheaves in characteristic zero. By constrast, we are unable to say much in positive characteristic. In fact,

a partial analogue of [Li, Thm. 4] has been proved over algebraic closures of finite fields [C, Thm. 1] (see also [D3]). This result could be used to prove a version of Theorem 7.2 only if Assumption 7.1 (b) were replaced by the following much more restrictive hypothesis: for all $\varphi \in \Phi_Z$, the strata Y_φ are spectra of finite products of fields, which are algebraic over some $\mathbb{F}_p$, for $p \neq \ell$.

(d) Assume the base scheme $\mathbb{B}$ to be arithmetic. Let $\pi : S(\mathfrak{S}) \to Y(\Phi)$ be a proper morphism of good stratifications satisfying Assumption 7.1, and such that the generic fiber $Y(\Phi) \otimes_{\mathbb{Z}} \mathbb{Q}$ is proper over $\mathbb{B} \otimes_{\mathbb{Z}} \mathbb{Q}$. Assume compatibility of the intermediate extension with the p-adic realization. One may then employ [KM, Thm. 2 2)] (*e.g.* as in the proof of [Sll, Thm. 1.2.4 (ii)]), to see that for primes p of good reduction, the characteristic polynomials of a Frobenius on p-adic intersection cohomology of $Y(\Phi) \otimes_{\mathbb{Z}} \mathbb{Q}$ and on the associated ϕ-filtered module coincide. This principle will be exploited elsewhere.

(e) At this point, the reader will have noted that the title of the present work is somewhat misleading. Indeed, our main results on the existence and the properties of the motivic intermediate extension (Corollary 5.7, Theorems 6.6, 6.7 and 7.2) only require $\pi_Z : S(\mathfrak{S}_Z) \to Y(\Phi_Z)$ to be of Abelian type. Our theory thus works for Chow motives "admitting a degeneration of Abelian type". In particular, it may be expected to apply to compactifications of moduli spaces of $K3$-surfaces.

8 Applications

Our main application of the material developed in the preceding sections concerns Shimura varieties and their compactifications. We shall recall the data necessary for their construction, and simultaneously check the validity of Assumptions 5.6, 6.5 and 7.1 (Lemmata 8.1–8.4).

Our notation is identical to the one used in [P1, P2, W1, BW]. Let $(P, \mathfrak{X})$ be *mixed Shimura data* [P1, Def. 2.1]. In particular, P is a connected algebraic linear group over $\mathbb{Q}$, and $P(\mathbb{R})$ acts on the complex manifold $\mathfrak{X}$ by analytic automorphisms. Denote by W the unipotent radical of P. If P is reductive, *i.e.*, if $W = 0$, then $(P, \mathfrak{X})$ is called *pure*. The *Shimura varieties* associated to $(P, \mathfrak{X})$ are indexed by the open compact subgroups of $P(\mathbb{A}_f)$. If K is one such, then the analytic space of $\mathbb{C}$-valued points of the corresponding variety is given as

$$M^K(\mathbb{C}) := P(\mathbb{Q}) \backslash (\mathfrak{X} \times P(\mathbb{A}_f)/K) ,$$

where $\mathbb{A}_f$ denotes the ring of finite adèles over $\mathbb{Q}$. According to Pink's generalization to mixed Shimura varieties of the Algebraization Theorem of Baily

and Borel [P1, Prop. 9.24], there exist canonical structures of normal quasi-projective varieties on the $M^K(\mathbb{C})$, which we denote as

$$M^K_{\mathbb{C}} := M^K(P, \mathfrak{X})_{\mathbb{C}} .$$

According to [M, Thm. 2.18] and [P1, Thm. 11.18], there is a *canonical model* of $M^K_{\mathbb{C}}$, which we denote as

$$M^K := M^K(P, \mathfrak{X}) .$$

It is defined over the *reflex field* $E(P, \mathfrak{X})$ of $(P, \mathfrak{X})$ [P1, 11.1].

Any *admissible parabolic subgroup* [P1, Def. 4.5] Q of P has a canonical normal subgroup P_1 [P1, 4.7]. There is a finite collection of *rational boundary components* $(P_1, \mathfrak{X}_1)$, indexed by the $P_1(\mathbb{R})$-orbits in $\pi_0(\mathfrak{X})$ [P1, 4.11]. The $(P_1, \mathfrak{X}_1)$ are themselves mixed Shimura data. Consider the following condition on $(P, \mathfrak{X})$ (cmp. [P2, Condition (3.1.5)]).

(+) If G denotes the maximal reductive quotient of P, then the neutral connected component $Z(G)^0$ of the center $Z(G)$ of G is, up to isogeny, a direct product of a $\mathbb{Q}$-split torus with a torus T of compact type (*i.e.*, $T(\mathbb{R})$ is compact) defined over $\mathbb{Q}$.

If $(P, \mathfrak{X})$ satisfies (+), then so does any rational boundary component $(P_1, \mathfrak{X}_1)$ [P1, proof of Cor. 4.10]. Similarly, the reflex field does not change when passing from $(P, \mathfrak{X})$ to a rational boundary component [P1, Prop. 12.1].

Fix a connected component $\mathfrak{X}^0$ of $\mathfrak{X}$, and denote by $(P_1, \mathfrak{X}_1)$ the associated rational boundary component. Denote by $U_1 \subset P_1$ the "weight -2" part of P_1. It is Abelian, normal in Q, and central in the unipotent radical W_1 of P_1. [P1, 4.11, 4.14, Prop. 4.15] give the construction of an open convex cone

$$C(\mathfrak{X}^0, P_1) \subset U_1(\mathbb{R})(-1)$$

canonically associated to P_1 and $\mathfrak{X}^0$.

Set-theoretically, the *conical complex* associated to $(P, \mathfrak{X})$ is defined as

$$\mathcal{C}(P, \mathfrak{X}) := \coprod_{(\mathfrak{X}^0, P_1)} C(\mathfrak{X}^0, P_1) .$$

By [P1, 4.24], the conical complex is naturally equipped with a topology (usually different from the coproduct topology). The closure $C^*(\mathfrak{X}^0, P_1)$ of $C(\mathfrak{X}^0, P_1)$ inside $\mathcal{C}(P, \mathfrak{X})$ can still be considered as a convex cone in $U_1(\mathbb{R})(-1)$, with the induced topology.

The *toroidal compactifications* of M^K are parameterized by *K-admissible complete cone decompositions*, which are collections of subsets of

$$\mathcal{C}(P, \mathfrak{X}) \times P(\mathbb{A}_f)$$

satisfying the axioms [P1, 6.4 (i'), (ii), (iii), (iv), (v)]. If $\mathfrak{S}$ is one such, then in particular any member of $\mathfrak{S}$ is of the shape

$$\sigma \times \{p\} ,$$

$p \in P(\mathbb{A}_f)$, $\sigma \subset C^*(\mathfrak{X}^0, P_1)$ a convex rational polyhedral cone in the vector space $U_1(\mathbb{R})(-1)$ not containing any non-trivial linear subspace.

Let $M^K(\mathfrak{S}) := M^K(P, \mathfrak{X}, \mathfrak{S})$ be the associated compactification; we refer to [P1, Thm. 9.21, 9.27, Prop. 9.28] for criteria sufficient to guarantee its existence. It comes equipped with a natural stratification into locally closed strata. Any such stratum is obtained as follows: fix a pair $(\mathfrak{X}^0, P_1)$ as above, $p \in P(\mathbb{A}_f)$ and

$$\sigma \times \{p\} \in \mathfrak{S}$$

such that $\sigma \subset C^*(\mathfrak{X}^0, P_1)$, $\sigma \cap C(\mathfrak{X}^0, P_1) \neq \emptyset$. To σ, one associates Shimura data

$$\big(P_{1,[\sigma]}, \mathfrak{X}_{1,[\sigma]}\big)$$

[P1, 7.1], whose underlying group $P_{1,[\sigma]}$ is the quotient of P_1 by the algebraic subgroup

$$\langle \sigma \rangle \subset U_1$$

satisfying $\mathbb{R} \cdot \sigma = \frac{1}{2\pi i} \cdot \langle \sigma \rangle(\mathbb{R})$. Set

$$K_1 := P_1(\mathbb{A}_f) \cap p \cdot K \cdot p^{-1} , \quad \pi_{[\sigma]} : P_1 \longrightarrow\!\!\!\!\rightarrow P_{1,[\sigma]} \ .$$

According to [P1, 7.3], there is a canonical map

$$i_{\sigma,p}(\mathbb{C}) : M^{\pi_{[\sigma]}(K_1)}(P_{1,[\sigma]}, \mathfrak{X}_{1,[\sigma]})(\mathbb{C}) \longrightarrow M^K(\mathfrak{S})(\mathbb{C}) := M^K(P, \mathfrak{X}, \mathfrak{S})(\mathbb{C})$$

whose image is locally closed. The latter is disjoint from $M^K(\mathbb{C})$ if and only if the admissible parabolic subgroup Q giving rise to P_1 is *proper*, i.e., unequal to P. $i_{\sigma,p}(\mathbb{C})$ comes from a morphism of schemes over $E(P, \mathfrak{X})$ [P1, Thm. 12.4 (a), (c)], denoted

$$i_{\sigma,p} : M^{\pi_{[\sigma]}(K_1)} := M^{\pi_{[\sigma]}(K_1)}(P_{1,[\sigma]}, \mathfrak{X}_{1,[\sigma]}) \longrightarrow M^K(\mathfrak{S}) \ .$$

Its image equals the stratum associated to $\sigma \times \{p\}$ Letting $\sigma \times \{p\}$ vary, the natural stratification

$$M^K(\mathfrak{S}) = \coprod_{\sigma \times \{p\}} i_{\sigma,p}(M^{\pi_{[\sigma]}(K_1)})$$

is indexed by a quotient of $\mathfrak{S}$, which is finite [P1, 7.3]. By abuse of notation, it will be denoted by the same letter $\mathfrak{S}$. Similarly, we shall write σ for the class of $\sigma \times \{p\}$. If $(P, \mathfrak{X})$ satisfies (+), and K is *neat* (see e.g. [P1, 0.6]), then $i_{\sigma,p}$ is an immersion [W1, Prop. 1.6], *i.e.*, it identifies $M^{\pi_{[\sigma]}(K_1)}$ with a

locally closed sub-scheme of $M^K(\mathfrak{S})$. In that case, the natural stratification takes the form

$$M^K(\mathfrak{S}) = \coprod_{\sigma\in\mathfrak{S}} M^{\pi_{[\sigma]}(K_1)} \ .$$

Lemma 8.1. *(a) The stratification* $M^K(\mathfrak{S}) = \coprod_{\sigma\in\mathfrak{S}} i_{\sigma,p}(M^{\pi_{[\sigma]}(K_1)})$ *is good,* i.e., *the closure* $\overline{i_{\sigma,p}(M^{\pi_{[\sigma]}(K_1)})}$ *of any stratum is a union of strata.*
(b) Assume that $(P,\mathfrak{X})$ *satisfies* $(+)$*, and that* K *is neat. Then modulo a possible replacement of* $\mathfrak{S}$ *by a suitable refinement,* $M^K(\mathfrak{S})$ *is a projective variety over* $E(P,\mathfrak{X})$*, and the closures* $\overline{M^{\pi_{[\sigma]}(K_1)}}$ *of strata,* $\sigma\in\mathfrak{S}$ *are smooth over* $E(P,\mathfrak{X})$*.*

Proof. By [P1, 7.11],

$$i_{\sigma,p}(\mathbb{C}) : M^{\pi_{[\sigma]}(K_1)}(\mathbb{C}) \longrightarrow M^K(\mathfrak{S})(\mathbb{C})$$

extends to a continuous map

$$\overline{i_{\sigma,p}}(\mathbb{C}) : M^{\pi_{[\sigma]}(K_1)}(\mathfrak{S}_{1,[\sigma]})(\mathbb{C}) \longrightarrow M^K(\mathfrak{S})(\mathbb{C})$$

of toroidal compactifications. Here, $\mathfrak{S}_{1,[\sigma]}$ is the cone decomposition defined in [P1, 7.7]. Since $\mathfrak{S}$ is assumed to be complete, the image of $\overline{i_{\sigma,p}}(\mathbb{C})$ equals the closure of $i_{\sigma,p}(M^{\pi_{[\sigma]}(K_1)})(\mathbb{C})$ [P1, Prop. 6.27, Prop. 7.8].

By construction [P1, 7.11], the extension $\overline{i_{\sigma,p}}(\mathbb{C})$ maps any stratum of $M^{\pi_{[\sigma]}(K_1)}(\mathfrak{S}_{1,[\sigma]})(\mathbb{C})$ onto a stratum of $M^K(\mathfrak{S})(\mathbb{C})$. This shows part (a), since the claim can be checked on $\mathbb{C}$-valued points.

Let us turn to the proof of part (b). According to [P1, proof of Thm. 9.21], a suitable refinement $\mathfrak{S}'$ of $\mathfrak{S}$ is smooth with respect to K [P1, 6.4], satisfies condition [P1, 7.12 ($*$)], and the associated toroidal compactification $M^K(\mathfrak{S}')$ (exists and) is projective over $E(P,\mathfrak{X})$. Let us change notation, and write $\mathfrak{S}$ instead of $\mathfrak{S}'$.

Together with condition $(+)$ and neatness of K, smoothness of $\mathfrak{S}$ implies that the cone decomposition $\mathfrak{S}_{1,[\sigma]}$ is smooth, too. Indeed, the groups $\Gamma_U \subset U_2(\mathbb{Q})$ defined in [P1, 6.4], for any boundary component $(P_2,\mathfrak{X}_2)$ of $(P,\mathfrak{X})$, and any element $p\in P(\mathbb{A}_f)$, are then equal to $U_2(\mathbb{Q})\cap pKp^{-1}$, *i.e.*, they do not change when passing from $\mathfrak{S}$ to the cone decomposition denoted

$$\mathfrak{S}_1 := \left([\ \cdot p]^*\mathfrak{S}\right)_{(P_1,\mathfrak{X}_1)}$$

in [P1, 7.3].

By [P1, Prop. 6.26], $M^{\pi_{[\sigma]}(K_1)}(\mathfrak{S}_{1,[\sigma]})(\mathbb{C})$ is smooth.

Thanks to condition [P1, 7.12 ($*$)], we may apply [P1, Cor. 7.17 (a)]. It tells us that the image of $\overline{i_{\sigma,p}}(\mathbb{C})$ equals the quotient of $M^{\pi_{[\sigma]}(K_1)}(\mathfrak{S}_{1,[\sigma]})(\mathbb{C})$ by the action of a certain group denoted $\mathrm{Stab}_{\Delta_1}([\sigma])$. But according to [W1, Lemma 1.7, Rem. 1.8], this group is trivial, thanks to condition $(+)$ and neatness of K. This establishes the claim, since smoothness of the $\overline{M^{\pi_{[\sigma]}(K_1)}}$ can be checked on $\mathbb{C}$-valued points. **q.e.d.**

Now let $(G, \mathfrak{H})$ be the pure Shimura data underlying $(P, \mathfrak{X})$, *i.e.*, let $(G, \mathfrak{H}) := (P, \mathfrak{X})/W$ be the quotient of $(P, \mathfrak{X})$ by W [P1, Prop. 2.9]. Fix an open compact subgroup $L \subset G(\mathbb{A}_f)$. Let $(M^L)^*$ denote the *Baily–Borel compactification* of $M^L := M^L(G, \mathfrak{H})$. It is a projective variety over the reflex field $E(G, \mathfrak{H}) = E(P, \mathfrak{X})$ [P1, Main Theorem 12.3 (a), (b)]. It comes equipped with a natural stratification into locally closed strata. Any stratum disjoint from M^L is obtained as follows: fix a proper admissible parabolic subgroup Q of G with associated normal subgroup P_1. Fix a rational boundary component $(P_1, \mathfrak{X}_1)$ of $(G, \mathfrak{H})$, and an element $g \in G(\mathbb{A}_f)$. Define

$$L_1 := P_1(\mathbb{A}_f) \cap g \cdot L \cdot g^{-1} .$$

Denote by W_1 the unipotent radical of P_1, and by

$$\pi_1 : (P_1, \mathfrak{X}_1) \longrightarrow\!\!\!\!\!\rightarrow (G_1, \mathfrak{H}_1) := (P_1, \mathfrak{X}_1)/W_1$$

the quotient of $(P_1, \mathfrak{X}_1)$ by W_1. According to [P1, 7.6, Main Theorem 12.3 (c)], there is a canonical morphism

$$i_g : M^{\pi_1(L_1)} := M^{\pi_1(L_1)}(G_1, \mathfrak{H}_1) \longrightarrow (M^L)^*$$

whose image is locally closed, and identical to the stratum associated to $(P_1, \mathfrak{X}_1)$ and g.

Letting $(P_1, \mathfrak{X}_1)$ and g vary, the natural stratification

$$(M^L)^* = \coprod_{(P_1, \mathfrak{X}_1), g} i_g(M^{\pi_1(L_1)})$$

is indexed by a quotient Φ of $\{(P_1, \mathfrak{X}_1)\} \times G(\mathbb{A}_f)$, which is finite [P1, 6.3].

Lemma 8.2. *(a) The stratification $(M^L)^* = \coprod_\Phi i_g(M^{\pi_1(L_1)})$ is good.*
(b) Assume that $(G, \mathfrak{H})$ satisfies $(+)$, and that L is neat. Then all strata $i_g(M^{\pi_1(L_1)})$, $((P_1, \mathfrak{X}_1), g) \in \Phi$ are smooth over $E(G, \mathfrak{H})$.

Proof. As in the proof of Lemma 8.1, the claims can be checked on the level of $\mathbb{C}$-valued points. By [P1, 7.6],

$$i_g(\mathbb{C}) : M^{\pi_1(L_1)}(\mathbb{C}) \longrightarrow (M^L)^*(\mathbb{C})$$

extends to a continuous map

$$\overline{i_g}(\mathbb{C}) : (M^{\pi_1(L_1)})^*(\mathbb{C}) \longrightarrow (M^L)^*(\mathbb{C})$$

of Baily–Borel compactifications. Since $(M^{\pi_1(L_1)})^*(\mathbb{C})$ is compact, the image of $\overline{i_g}(\mathbb{C})$ equals the closure of $i_g(M^{\pi_1(L_1)})(\mathbb{C})$. By construction [P1, 7.6], the extension $\overline{i_g}(\mathbb{C})$ maps any stratum of $(M^{\pi_1(L_1)})^*(\mathbb{C})$ onto a stratum of $(M^L)^*(\mathbb{C})$. This shows part (a).

In order to prove part (b), define the following group (cmp. [BW, Sect. 1]):

$$H_Q := \mathrm{Stab}_{Q(\mathbb{Q})}(\mathfrak{H}_1) \cap P_1(\mathbb{A}_f) \cdot L' .$$

H_Q acts by analytic automorphisms on $\mathfrak{H}_1 \times P_1(\mathbb{A}_f)/L_1$. Hence the group $\Delta_1 := H_Q/P_1(\mathbb{Q})$ acts naturally on

$$M^{\pi_1(L_1)}(\mathbb{C}) = P_1(\mathbb{Q})\backslash(\mathfrak{H}_1 \times P_1(\mathbb{A}_f)/L_1) \ .$$

By [P1, 6.3], the action of Δ_1 factors through a finite quotient Δ, and the quotient by this action is precisely the image of $i_g(\mathbb{C})$. By [BW, Prop. 1.1 (d)], the action of Δ on $M^{\pi_1(L_1)}(\mathbb{C})$ is free provided that $(G, \mathfrak{H})$ satisfies $(+)$, and L is neat. But $M^{\pi_1(L_1)}(\mathbb{C})$ is smooth: indeed, L being neat, so is $\pi(L_1)$. Now apply [P1, Prop. 3.3 (b)]. **q.e.d.**

We need to analyze morphisms associated to change of the level structure L, and their extension to the Baily-Borel compactifications. Thus, let L' be a second open compact subgroup of $G(\mathbb{A}_f)$, and $h \in G(\mathbb{A}_f)$ such that $L \subset h \cdot L' \cdot h^{-1}$. Then there is a finite, surjective morphism

$$[\ \cdot h] : M^L \longrightarrow M^{L'}$$

[P1, 3.4 (a), Def. 11.5]. It extends to a finite, surjective morphism

$$(M^L)^* \longrightarrow (M^{L'})^*$$

[P1, Main Theorem 12.3 (b)], which we still denote by $[\ \cdot h]$. By the description from [P1, 6.3], one sees that $[\ \cdot h] : (M^L)^* \to (M^{L'})^*$ is a morphism of stratifications with respect to the natural stratifications

$$(M^L)^* = \coprod_{\Phi} i_g(M^{\pi_1(L_1)}) \quad \text{and} \quad (M^{L'})^* = \coprod_{\Phi} i_g(M^{\pi_1(L'_1)})$$

from above. Fix $M^{\pi_1(L_1)}$ and $M^{\pi_1(L'_1)}$ occurring in these stratifications.

Lemma 8.3. *Assume that $(G, \mathfrak{H})$ satisfies $(+)$, that L' is neat (hence so is L), and that $i_g(M^{\pi_1(L_1)})$ is a stratum of $[\ \cdot h]^{-1}(i_g(M^{\pi_1(L'_1)})$. Then the morphism induced by $[\ \cdot h]$ on $i_g(M^{\pi_1(L_1)})$,*

$$[\ \cdot h]_g : i_g(M^{\pi_1(L_1)}) \longrightarrow i_g(M^{\pi_1(L'_1)})$$

is finite and étale.

Proof. The varieties $M^{\pi_1(L_1)}$ and $M^{\pi_1(L'_1)}$ are associated to the same Shimura data $(G_1, \mathfrak{H}_1)$. From the description of the morphisms i_g on complex points [P1, 6.3], it follows that $[\ \cdot h]_g$ fits into a commutative diagram

$$\begin{tikzcd}
M^{\tilde{L}_1} := M^{\tilde{L}_1}(G_1, \mathfrak{H}_1) \arrow[r, two heads, "b'"] \arrow[d, "{[\ \cdot h_1]}"'] & \Delta\backslash M^{\tilde{L}_1} \cong i_g(M^{\pi_1(L_1)}) \arrow[d, "{[\ \cdot h]_g}"] \\
M^{\tilde{L'}_1} := M^{\tilde{L'}_1}(G_1, \mathfrak{H}_1) \arrow[r, two heads, "b"] & \Delta'\backslash M^{\tilde{L'}_1} \cong i_g(M^{\pi_1(L'_1)})
\end{tikzcd}$$

for suitable open compact subgroups $\tilde{L}_1$ and $\tilde{L'}_1$ of $G_1(\mathbb{A}_f)$, which are neat, an element h_1 of $G_1(\mathbb{A}_f)$ such that $\tilde{L}_1 \subset h_1 \cdot \tilde{L'}_1 \cdot h_1^{-1}$, and finite groups Δ and Δ' acting on $M^{\tilde{L}_1}$ and $M^{\tilde{L'}_1}$, respectively (see the proof of Lemma 8.2).

Their actions being free, the surjective morphisms b and b' are finite and étale. But so is the morphism $[\ \cdot h_1]$. **q.e.d.**

Let us now connect the two types of compactifications. Let $(P, \mathfrak{X})$ be mixed Shimura data as before,

$$\pi : (P, \mathfrak{X}) \longrightarrow\!\!\!\!\rightarrow (G, \mathfrak{H}) = (P, \mathfrak{X})/W \ ,$$

K an open compact subgroup of $P(\mathbb{A}_f)$. Put $L := \pi(K)$. The morphism of Shimura data π gives rise to a surjective morphism

$$M^K = M^K(P, \mathfrak{X}) \longrightarrow M^L = M^L(G, \mathfrak{H})$$

[P1, 3.4 (b), Prop. 11.10], equally denoted π. Let $\mathfrak{S}$ be a K-admissible complete cone decomposition. Then π extends to a proper, surjective morphism

$$M^K(\mathfrak{S}) \longrightarrow (M^L)^*$$

[P1, 6.24, Main Theorem 12.4 (b)], still denoted π. From the description given in [P1, 7.3], one sees that $\pi : M^K(\mathfrak{S}) \to (M^L)^*$ is a morphism of stratifications with respect to the natural stratifications

$$M^K(\mathfrak{S}) = \coprod_{\sigma \in \mathfrak{S}} i_{\sigma,p}(M^{\pi_{[\sigma]}(K_1)}) \quad \text{and} \quad (M^L)^* = \coprod_{\Phi} i_g(M^{\pi_1(L_1)})$$

from above. Fix $M^{\pi_{[\sigma]}(K_1)}$ and $M^{\pi_1(L_1)}$ occurring in these stratifications.

Lemma 8.4. *Assume that $(P, \mathfrak{X})$ satisfies $(+)$, that K is neat, and that $M^{\pi_{[\sigma]}(K_1)}$ is a stratum of $\pi^{-1}(i_g(M^{\pi_1(L_1)}))$. Then the morphism induced by π on $M^{\pi_{[\sigma]}(K_1)}$,*

$$\pi_\sigma : M^{\pi_{[\sigma]}(K_1)} \longrightarrow i_g(M^{\pi_1(L_1)})$$

can be factorized

$$\pi_\sigma = \pi'_\sigma \circ \pi''_\sigma : M^{\pi_{[\sigma]}(K_1)} \stackrel{\pi''_\sigma}{\longrightarrow} B_\sigma \stackrel{\pi'_\sigma}{\longrightarrow} i_g(M^{\pi_1(L_1)}) \ ,$$

such that the morphisms π'_σ and π''_σ satisfy the following conditions.

(1) *π'_σ is proper and smooth, and its pull-back to any geometric point of $i_g(M^{\pi_1(L_1)})$ is isomorphic to a finite disjoint union of Abelian varieties.*

(2) *The motive*

$$\pi''_{\sigma,*} \mathbb{1}_{M^{\pi_{[\sigma]}(K_1)}} \in DM_{\text{Б},c}(B_\sigma)_F$$

belongs to the category $DMT(B_\sigma)_F$ of Tate motives over B_σ.

Proof. The varieties $M^{\pi_1(L_1)}$ and $M^{\pi_{[\sigma]}(K_1)}$ are associated to Shimura data $(G_1, \mathfrak{H}_1)$ and $(P_{1,[\sigma]}, \mathfrak{X}_{1,[\sigma]})$, respectively. Since $M^{\pi_{[\sigma]}(K_1)}$ belongs to $\pi^{-1}(i_g(M^{\pi_1(L_1)}))$, the Shimura data $(G_1, \mathfrak{H}_1)$ are in fact the pure Shimura data underlying $(P_{1,[\sigma]}, \mathfrak{X}_{1,[\sigma]})$. From the description of the morphisms i_g and

$i_{\sigma,p}$ on complex points [P1, 6.3, 7.3], it follows that π_σ is isomorphic to the composition

$$M^{K_{1,[\sigma]}} := M^{K_{1,[\sigma]}}(P_{1,[\sigma]}, \mathfrak{X}_{1,[\sigma]}) \xrightarrow{\pi_1} M^{\pi_1(K_{1,[\sigma]})} \xrightarrow{b} \Delta\backslash M^{\pi_1(K_{1,[\sigma]})} ,$$

where we let $\pi_1 : (P_{1,[\sigma]}, \mathfrak{X}_{1,[\sigma]}) \longrightarrow\!\!\!\!\rightarrow (G_1, \mathfrak{H}_1)$ denote the quotient map, for a suitable open compact subgroup $K_{1,[\sigma]}$ of $P_{1,[\sigma]}(\mathbb{A}_f)$, which is neat, and for a finite group Δ acting on $M^{\pi_1(K_{1,[\sigma]})} := M^{\pi_1(K_{1,[\sigma]})}(G_1, \mathfrak{H}_1)$ (see the proof of Lemma 8.2). The action of Δ being free, the morphism b is finite and étale. The factorization of $\pi_1 : (P_{1,[\sigma]}, \mathfrak{X}_{1,[\sigma]}) \longrightarrow\!\!\!\!\rightarrow (G_1, \mathfrak{H}_1)$ corresponding to the weight filtration $1 \subset U_{1,[\sigma]} \subset W_{1,[\sigma]}$ of the unipotent radical $W_{1,[\sigma]}$ of $P_{1,[\sigma]}$ gives the following:

$$\begin{tikzcd} (P_{1,[\sigma]}, \mathfrak{X}_{1,[\sigma]}) \arrow[r, two heads, "\pi_t"] \arrow[rr, bend right, "\pi_1"'] & (P'_1, \mathfrak{X}'_1) := (P_{1,[\sigma]}, \mathfrak{X}_{1,[\sigma]}) / U_{1,[\sigma]} \arrow[r, two heads, "\pi_a"] & (G_1, \mathfrak{H}_1) \end{tikzcd}$$

On the level of Shimura varieties, we get:

$$\begin{tikzcd} M^{K_{1,[\sigma]}} \arrow[r, two heads, "\pi_t"] \arrow[rr, bend right, "\pi_1"'] & M^{\pi_t(K_{1,[\sigma]})}(P'_1, \mathfrak{X}'_1) \arrow[r, two heads, "\pi_a"] & M^{\pi_1(K_{1,[\sigma]})} \end{tikzcd}$$

By [P1, 3.12–3.22 (a), Prop. 11.10], π_a is in a natural way a torsor under an Abelian scheme, while π_t is a torsor under a split torus T. Now put

$$B_\sigma := M^{\pi_t(K_{1,[\sigma]})}(P'_1, \mathfrak{X}'_1) \quad , \quad \pi''_\sigma := \pi_t \quad \text{and} \quad \pi'_\sigma := b \circ \pi_a .$$

q.e.d.

The verification of Assumptions 5.6, 6.5 and 7.1 is now complete. The theory of the intermediate extension therefore applies. In order to make it explicit, assume that $(P, \mathfrak{X})$ satisfies (+), and that K is neat. As before, denote by π the morphism $M^K(\mathfrak{S}) \to (M^L)^*$. Let $U \subset (M^L)^*$ be an open union of strata, and denote by $\mathfrak{S}_U \subset \mathfrak{S}$ the set of strata of $M^K(\mathfrak{S})$ mapping to U under π. Assume that $\mathfrak{S}$ is fine enough for Lemma 8.1 (b) to hold.

Theorem 8.5. *The hypotheses of Theorem 2.9 (a) are fulfilled for* $\mathcal{C}(X) = \pi_* DMT_{\mathfrak{S}}(M^K(\mathfrak{S}))_F$ *and* $\mathcal{C}(U) = \pi_* DMT_{\mathfrak{S}_U}(\pi^{-1}(U))_F$. *In particular, the intermediate extension*

$$j_{!*} : \pi_* DMT_{\mathfrak{S}_U}\big(\pi^{-1}(U)\big)_{F,w=0} \hookrightarrow \pi_* DMT_{\mathfrak{S}}\big(M^K(\mathfrak{S})\big)^u_{F,w=0}$$

is defined, and it satisfies the properties listed in Summary 2.12.

Proof. The morphisms π is proper and respects the stratifications, which are good according to Lemmata 8.1 (a) and 8.2 (a). Assumption 5.6 (a) holds thanks to Lemma 8.1 (b). So do Assumption 5.6 (b) (by Lemma 8.2 (b)) and Assumption 5.6 (c) (by Lemma 8.4 and Example 5.9 (c)).

Now apply Corollary 5.7. **q.e.d.**

Proof of Theorem 0.1. This is the special case of a trivial "weight -2" part of P, and of $U = M^L \subset (M^L)^*$. According to Theorem 8.5,

$$j_{!*} : \pi_* DMT\big(M^K\big)_{F,w=0} \hookrightarrow \pi_* DMT_{\mathfrak{S}}\big(M^K(\mathfrak{S})\big)^u_{F,w=0}$$

is defined for a suitable choice of $\mathfrak{S}$. Theorem 0.1 concerns the Tate motive $\mathbb{1}_{M^K}$, and direct factors N of its direct image

$$\pi_* \mathbb{1}_{M^K} \in \pi_* DMT\big(M^K\big)_{F,w=0} \,.$$

For $N = \pi_* \mathbb{1}_{M^K}$, all claims are contained in Summary 2.12. In particular (Theorem 0.1 (d)), any idempotent endomorphism of $\pi_* \mathbb{1}_{M^K}$ extends idempotently to $j_{!*} \pi_* \mathbb{1}_{M^K}$. It follows formally that Theorem 0.1 holds for arbitrary direct factors of $\pi_* \mathbb{1}_{M^K}$. **q.e.d.**

Theorem 8.6. *The intermediate extension is compatible with the Betti realization; more precisely, the following holds.*
(a) For any integer m, the restriction of the composition

$$H^m \circ R_{(M^L)^*} : DM_{\text{Б},c}\big((M^L)^*\big)_F \longrightarrow \mathbf{Perv}_c\big((M^L)^*(\mathbb{C}), F\big)$$

to $\pi_* DMT_{\mathfrak{S}}(M^K(\mathfrak{S}))_{F,w=0}$ factors over $\pi_* DMT_{\mathfrak{S}}(M^K(\mathfrak{S}))^u_{F,w=0}$.
(b) For any integer m, the diagram

$$\begin{tikzcd}
\pi_* DMT_{\mathfrak{S}_U}\big(\pi^{-1}(U)\big)_{F,w=0} \arrow[r, hook, "j_{!*}"] \arrow[d, "H^m \circ R_U"'] & \pi_* DMT_{\mathfrak{S}}\big(M^K(\mathfrak{S})\big)^u_{F,w=0} \arrow[d, "H^m \circ R_{(M^L)^*}"] \\
\mathbf{Perv}_c(U(\mathbb{C}), F) \arrow[r, hook, "j_{!*}"] & \mathbf{Perv}_c\big((M^L)^*(\mathbb{C}), F\big)
\end{tikzcd}$$

commutes.

Proof. Assumption 7.1 (a) and (b) hold thanks to Lemmata 8.1 and 8.2. So does Assumption 7.1 (c) (by Lemma 8.4).
Now apply Theorem 7.2. **q.e.d.**

Proof of Theorem 0.2. Theorem 8.6 covers the case of

$$\pi_* \mathbb{1}_{M^K} \in \pi_* DMT\big(M^K\big)_{F,w=0} \,.$$

The isomorphisms

$$H^m \circ R_{(M^L)^*} \circ j_{!*} \cong j_{!*} \circ H^m \circ R_{M^L}$$

from (b) being isomorphisms of additive functors, they extend to arbitrary direct factors N of $\pi_* \mathbb{1}_{M^K}$. **q.e.d.**

Now let L' be a second open compact subgroup of $G(\mathbb{A}_f)$, $h \in G(\mathbb{A}_f)$ such that $L \subset h \cdot L' \cdot h^{-1}$, and

$$[\,\cdot h] : (M^L)^* \longrightarrow (M^{L'})^*$$

as before. Assume that the open subscheme U of $(M^L)^*$ equals $[\,\cdot h]^{-1}(U')$, for some open subscheme U' of $(M^{L'})^*$. Denote by Z the complement of U in $(M^L)^*$.

Theorem 8.7. *The intermediate extension is compatible with the direct image* $[\,\cdot h]_*$*; more precisely, the following holds.*
(a) The functor

$$[\,\cdot h]_* : \pi_* DMT_{\mathfrak{S}}\big(\pi^{-1}(Z)\big)_{F,w=0} \longrightarrow \big([\,\cdot h] \circ \pi\big)_* DMT_{\mathfrak{S}}\big(\pi^{-1}(Z)\big)_{F,w=0}$$

is radicial.
(b) The diagram

$$\begin{array}{ccc} \pi_* DMT_{\mathfrak{S}_U}\big(\pi^{-1}(U)\big)_{F,w=0} & \xhookrightarrow{\;j_{!*}\;} & \pi_* DMT_{\mathfrak{S}}\big(M^K(\mathfrak{S})\big)^u_{F,w=0} \\ \Big\downarrow{\scriptstyle [\,\cdot h]_*} & & \Big\downarrow{\scriptstyle [\,\cdot h]^u_*} \\ \big([\,\cdot h] \circ \pi\big)_* DMT_{\mathfrak{S}_U}\big(\pi^{-1}(U)\big)_{F,w=0} & \xhookrightarrow{\;j_{!*}\;} & \big([\,\cdot h] \circ \pi\big)_* DMT_{\mathfrak{S}}\big(M^K(\mathfrak{S})\big)^u_{F,w=0} \end{array}$$

commutes.

Proof. Assumption 6.5 (a), (b) and (d) hold thanks to Lemmata 8.1, 8.2 and 8.4. So does Assumption 6.5 (c) (by Lemma 8.3).
Now apply Theorem 6.6. **q.e.d.**

Proof of Theorem 0.4. Theorem 8.7 covers the case of

$$\pi_* \mathbb{1}_{M^K} \in \pi_* DMT\big(M^K\big)_{F,w=0} .$$

For arbitrary direct factors N of $\pi_* \mathbb{1}_{M^K}$, the claim follows again from the fact (Theorem 0.1 (d)) that any idempotent endomorphism of $\pi_* \mathbb{1}_{M^K}$ can be extended to an idempotent endomorphism of $j_{!*}\, \pi_* \mathbb{1}_{M^K}$. **q.e.d.**

Let us conclude with a discussion of Hecke operators. We keep the situation of Theorems 8.5–8.7. In particular, we continue to assume that $(P, \mathfrak{X})$ satisfies (+), and that K is neat. In addition, we impose two more conditions. First, we assume that the "weight -2"par of P is trivial, *i.e.*, that the unipotent radical W is pure of weight -1. Second, we fix a Levi section $i : G \hookrightarrow P$ of the projection from P to G, and suppose that K contains the image of L under i. In other words, K is the semi-direct product of $K \cap W(\mathbb{A}_f)$ and $i(L)$. According to [P1, 3.12–3.22 (a), Prop. 11.10], the morphism $\pi : M^K \to M^L$ is then equipped with the structure of an Abelian scheme. We are thus in the situation of the Introduction. As there, we fix a direct factor N of

$$\pi_* \mathbb{1}_{M^K} \in CHM^s(M^L)_F \ .$$

Let us recall the action of the *Hecke algebra* $R(L, G(\mathbb{A}_f))$ associated to the Shimura variety M^L. Elements of $R(L, G(\mathbb{A}_f))$ are formal linear combinations of double cosets LxL, for $x \in G(\mathbb{A}_f)$. Fix such an x, and define

$L_x := L \cap x^{-1}Lx$. According to what was recalled further above, there are two finite, étale morphisms

$$g_1 := [\,\cdot 1]\ ,\ g_2 := [\,\cdot x^{-1}] : M^{L_x} \longrightarrow M^L\ .$$

In order to construct the action of LxL, we need three ingredients: (1) the adjunction morphism $adj_1 : N \to g_{1,*}g_1^*N$, (2) the adjunction morphism $adj_2 : g_{2,*}g_2^*N = g_{2,!}g_2^!N \to N$, (3) a canonical morphism $\varphi_x : g_1^*N \to g_2^*N$ of smooth Chow motives over L_x, whose construction we recall now. By smooth (or proper) base change [CD, Thm. 2.4.50 (4)],

$$g_r^*\pi_*\mathbb{1}_{M^K} = \pi_{r,*}\mathbb{1}_{M^{K_r}}\ ,\ r = 1, 2\ ,$$

where $\pi_r : M^{K_r} := M^{K_r}(P, \mathfrak{X}) \to M^{L_x}$, $r = 1, 2$ are the projections associated to $K_1 := K \cap \pi^{-1}(L_x)$ and $K_2 := y^{-1}Ky \cap \pi^{-1}(L_x)$, for $y := i(x)$ (recall that i is the Levi section fixed above). Then π_1 and π_2 are simultaneously dominated by $\pi_{1,2} : M^{K_{1,2}} := M^{K_{1,2}}(P, \mathfrak{X}) \to M^{L_x}$, for $K_{1,2} := K \cap y^{-1}Ky \cap \pi^{-1}(L_x)$; more precisely, $\pi_{1,2}$ factors over both M^{K_r} *via* finite, étale morphisms $M^{K_{1,2}} \to M^{K_r}$, $r = 1, 2$. Consider the composition of the adjunctions associated to these morphisms

$$\psi_x : g_1^*\pi_*\mathbb{1}_{M^K} = \pi_{1,*}\mathbb{1}_{M^{K_1}} \longrightarrow \pi_{1,2,*}\mathbb{1}_{M^{K_{1,2}}} \longrightarrow \pi_{2,*}\mathbb{1}_{M^{K_2}} = g_2^*\pi_*\mathbb{1}_{M^K}\ .$$

The desired morphism φ_x is the composition of the inclusion of g_1^*N into $g_1^*\pi_*\mathbb{1}_{M^K}$, of ψ_x, and of the projection of $g_2^*\pi_*\mathbb{1}_{M^K}$ onto g_2^*N.

Denote by m° the structure morphism $M^L \to \mathbf{Spec}\, E$. Given (1), (2), and (3), we may now define the endomorphism LxL of m°_*N. Namely, apply m°_* to (1) and (2),

$$m^\circ_*(adj_1) : m^\circ_*N \longrightarrow (m^\circ \circ g_1)_*g_1^*N\ ,\ m^\circ_*(adj_2) : (m^\circ \circ g_2)_*g_2^*N \longrightarrow m^\circ_*N\ ,$$

and $(m^\circ \circ g_1)_* = (m^\circ \circ g_2)_*$ to (3),

$$(m^\circ \circ g_1)_*(\varphi_x) : (m^\circ \circ g_1)_*g_1^*N \longrightarrow (m^\circ \circ g_2)_*g_2^*N\ .$$

Set

$$LxL := m^\circ_*(adj_2) \circ (m^\circ \circ g_1)_*(\varphi_x) \circ m^\circ_*(adj_1) : m^\circ_*N \longrightarrow m^\circ_*N\ .$$

In the same way, one gets an endomorphism of $m^\circ_!N$, which is denoted by the same symbol LxL. Denote the structure morphism $(M^L)^* \to \mathbf{Spec}\, E$ by m; thus,

$$m^\circ_*N = m_*j_*N \quad \text{and} \quad m^\circ_!N = m_*j_!N\ .$$

Denote the extension of g_r to a morphism $(M^{L_x})^* \to (M^L)^*$ [P1, Main Theorem 12.3 (b)] by the same letter g_r, $r = 1, 2$.

Corollary 8.8. *(a) The action of LxL can be extended to $m_* \, j_{!*} \, N$, in a way compatible with the actions on $m_* j_! N$ and on $m_* j_* N$: there is a commutative diagram*

$$\begin{tikzcd} m_* j_! N \arrow[r] \arrow[d, "LxL"'] & m_* \, j_{!*} \, N \arrow[r] \arrow[d, "LxL"'] & m_* j_* N \arrow[d, "LxL"] \\ m_* j_! N \arrow[r] & m_* \, j_{!*} \, N \arrow[r] & m_* j_* N \end{tikzcd}$$

(b) Assume that one of the following additional conditions (i), (ii) are satisfied: (i) the radicals

$$\mathrm{rad}_{CHM((M^L)^*)_F}(j_{!*} \, N, g_{1,*} \, j_{!*} \, g_2^* N) \subset \mathrm{Hom}_{CHM((M^L)^*)_F}(j_{!*} \, N, g_{1,*} \, j_{!*} \, g_2^* N)$$

and

$$\mathrm{rad}_{CHM((M^L)^*)_F}(g_{2,*} \, j_{!*} \, g_2^* N, j_{!*} \, N) \subset \mathrm{Hom}_{CHM((M^L)^*)_F}(g_{2,*} \, j_{!*} \, g_2^* N, j_{!*} \, N)$$

are trivial, (ii) the radicals

$$\mathrm{rad}_{CHM((M^L)^*)_F}(j_{!*} \, N, g_{1,*} \, j_{!*} \, g_1^* N) \subset \mathrm{Hom}_{CHM((M^L)^*)_F}(j_{!*} \, N, g_{1,*} \, j_{!*} \, g_1^* N)$$

and

$$\mathrm{rad}_{CHM((M^L)^*)_F}(g_{2,*} \, j_{!*} \, g_1^* N, j_{!*} \, N) \subset \mathrm{Hom}_{CHM((M^L)^*)_F}(g_{2,*} \, j_{!*} \, g_1^* N, j_{!*} \, N)$$

are trivial. Then the action of LxL on $m_* \, j_{!*} \, N$ is canonical.
(c) Assume that the Betti realization $R_{M^L}(N)$ is concentrated in a single perverse degree. Then the endomorphism

$$R_{\mathbf{Spec}\, E}(LxL) : R_{\mathbf{Spec}\, E}(m_* \, j_{!*} \, N) \longrightarrow R_{\mathbf{Spec}\, E}(m_* \, j_{!*} \, N)$$

coincides with the Hecke operator defined on the complex computing intersection cohomology via *the isomorphism*

$$R_{\mathbf{Spec}\, E}(m_* \, j_{!*} \, N) \cong m_* \, j_{!*} \, R_{M^L}(N)$$

from Corollary 0.3.

Remark 8.9. (a) One may expect the additional hypothesis of 8.8 (b) to be met as soon as N is indecomposable (cmp. Summary 2.12 (c), Conjecture 3.3).
(b) Recall that the motive $\pi_* \mathbb{1}_{M^K}$ admits a Chow–Künneth decomposition [DM, Thm. 3.1]. Any direct factor N contained in a Chow–Künneth component (in particular, any indecomposable N) then satisfies the additional hypothesis of 8.8 (c).
(c) Assume that the Shimura data $(G, \mathfrak{H})$ are of *PEL-type*. According to the main result from [Ac], the isomorphism classes of indecomposable direct factors of $\pi_* \mathbb{1}_{M^K}$ are in bijective correspondence with certain of the isomorphisms classes of simple algebraic representations of G. More precisely, the Chow motive $\pi_* \mathbb{1}_{M^K}$ corresponds to a direct sum V of simple representations of G, and any simple constituent V' of V lifts to a direct factor N of $\pi_* \mathbb{1}_{M^K}$, uniquely up to isomorphism [Ac, Thm. 4.7].

Proof of Corollary 8.8. Fix a $K_{1,2}$-admissible complete cone decomposition $\mathfrak{S}$ satisfying Lemma 8.1 (b). Applying $j_{!*}$ to (1) and (2), we get morphisms

$$j_{!*}(adj_1) : j_{!*} N \longrightarrow j_{!*} g_{1,*} g_1^* N \quad \text{and} \quad j_{!*}(adj_2) : j_{!*} g_{2,*} g_2^* N \longrightarrow j_{!*} N$$

in the quotient category $(g_1 \circ \pi_{1,2,*} \coprod g_2 \circ \pi_{1,2,*}) DMT_{\mathfrak{S}}(M^{K_{1,2}}(\mathfrak{S}))^u_{F,w=0}$. By Theorem 8.7 (b),

$$j_{!*} g_{r,*} g_r^* N = g^u_{r,*} j_{!*} g_r^* N \ , \ r = 1, 2 \ .$$

Applying $j_{!*}$ to (3), we get

$$j_{!*}(\varphi_x) : j_{!*} g_1^* N \longrightarrow j_{!*} g_2^* N$$

in $\pi_{1,2,*} DMT_{\mathfrak{S}}(M^K(\mathfrak{S}))^u_{F,w=0}$, yielding

$$g^u_{1,*} j_{!*}(\varphi_x) : g^u_{1,*} j_{!*} g_1^* N \longrightarrow g^u_{1,*} j_{!*} g_2^* N$$

in $(g_1 \circ \pi_{1,2,*} \coprod g_2 \circ \pi_{1,2,*}) DMT_{\mathfrak{S}}(M^{K_{1,2}}(\mathfrak{S}))^u_{F,w=0}$. Composing the latter with $j_{!*}(adj_1)$, we get

$$g^u_{1,*} j_{!*}(\varphi_x) \circ j_{!*}(adj_1) : j_{!*} N \longrightarrow g^u_{1,*} j_{!*} g_2^* N \ .$$

Lift both this latter morphism and $j_{!*}(adj_2)$ to actual morphisms

$$j_{!*} N \longrightarrow g_{1,*} j_{!*} g_2^* N \quad \text{and} \quad g_{2,*} j_{!*} g_2^* N \longrightarrow j_{!*} N$$

of Chow motives over $(M^L)^*$. According to Theorems 8.7 (b) and 1.9 (b), such lifts are unique up to morphisms belonging to the radical. Applying m_*, we get

$$m_* j_{!*} N \longrightarrow (m \circ g_1)_* j_{!*} g_2^* N \quad \text{and} \quad (m \circ g_2)_* j_{!*} g_2^* N \longrightarrow m_* j_{!*} N \ .$$

But since $m \circ g_1 = m \circ g_2$, the two morphisms can be composed; this is the desired endomorphism

$$LxL : m_* j_{!*} N \longrightarrow m_* j_{!*} N \ .$$

This proves parts (a) and (b). Part (c) follows from Theorem 8.6 (a) and Corollary 0.3. **q.e.d.**

Proof of Theorem 0.5. Theorem 0.5 is identical to parts (a) and (c) of Corollary 8.8. **q.e.d.**

Remark 8.10. For specific Shimura varieties, certain of the above results, or variants thereof, are already known.

(a) In [Sll, Sect. 1], the construction of the intersection motive $m_* j_{!*} N$ of modular curves with coefficients in symmetric powers N of the relative "h^1" of the universal elliptic curve is given. It carries an action of Hecke operators, and satisfies the properties from Theorem 0.5 [Sll, Prop. 4.1.3]. Actually, all the ingredients to define the intermediate extension $j_{!*} N$ itself can be found in [loc. cit.]— except for the notion of Chow motive over $(M^L)^*$. Scholl then

proceeds and uses the Hecke action to construct (Grothendieck) motives associated to modular forms.

(b) The main result of [GHM] applies to Hilbert–Blumenthal varieties $M^L_{\mathbb{C}}$ over $\mathbb{C}$. Indeed, according to [GHM, Thm I], for any Kuga family $\pi : M^K \to M^L$, any direct factor N of $\pi_* \mathbb{1}_{M^K_{\mathbb{C}}}$ admits an extension to a Chow motive on $(M^L)^*_{\mathbb{C}}$ satisfying Theorem 0.1 (a) (1); according to Remark 6.8, it is thus isomorphic to the base change of the intermediate extension of N to $\mathbb{C}$. Compatibility with the Betti realization (Theorem 0.2) is also established in [GHM, Thm I].

(c) The intersection motive of a surface S (or more generally, of a variety admitting a *semismall resolution*) over $\mathbb{C}$, with constant coeffients, is constructed in [CM, Thm. 4.0.4]. It satisfies Theorem 0.2 [CM, Rem. 4.0.6]. For surfaces, all the ingredients to define the intermediate extension itself can actually be found in [loc. cit.]. This construction even works over arbitrary fields. For details, we refer to [W4, Thm. 3.11 (b)].

(d) By [MS, Thm. 1], any direct factor N of $\pi_* \mathbb{1}_{M^K_{\mathbb{C}}}$ admits an extension to a Chow motive on $(M^L)^*_{\mathbb{C}}$ satisfying Theorem 0.2, as soon as M^K is of dimension 3.

(e) Let M^L be a Hilbert–Blumenthal variety, and N a direct factor of $\pi_* \mathbb{1}_{M^K}$ not containing any shift of a Tate twist. Then according to [W3, Cor. 2.7], the *interior motive* of M^L with coefficients in N (which in this geometric setting coincides with the intersection motive) exists. Theorem 0.5 is proved in [W3, Cor. 2.8], and Theorem 0.2 follows from [W2, Thm. 4.7].

(f) To the best of the author's knowlegde, the first non-trivial example of a candidate for the motivic intermediate extension over a Baily–Borel compactification of dimension at least 3, and defined over the reflex field of the Shimura data, appears in Vaish's thesis [V]. There, the author shows the existence of a motive EM_X, which is a natural candidate for a pre-image of the *weight truncated complex* EC_X [NV1] under the Betti realization [V, Thm. 2.3.30]. He also studies the relation of EC_X and the intersection complex IC_X. According to [V, Ex. 1.5.7], $EC_X = IC_X$ if X is the Baily–Borel compactification of a Hilbert–Blumenthal variety. This concerns the case of constant coefficients that was left open in [W3].

(g) In [NV2], the authors consider the case of Siegel threefolds M^L, and the universal family $\pi : M^K \to M^L$ of Abelian surfaces with an n-structure, for some $n \geq 3$. [NV2, Thm. I, Thm. II] establish Theorem 0.1 (a) and (d), for $N = \pi_* \mathbb{1}_{M^K}$. Contrary to our present approach, the construction of [NV2] is very explicit. For a particular choice of $\mathfrak{S}$, it is shown that the action of the n-torsion extends to $M^K(\mathfrak{S})$. Writing $\pi : M^K(\mathfrak{S}) \to (M^L)^*$ as before, it is then established that the direct factor of $\pi_* \mathbb{1}_{M^K(\mathfrak{S})}$ on which the n-torsion acts trivially, has no components supported in the boundary of $(M^L)^*$. Therefore, it is an intermediate extension of $\pi_* \mathbb{1}_{M^K}$. The claim from Theorem 0.1 (a) (2) is verified by using intersection theory of cycles on

$M^K(\mathfrak{S})$.

Remark 8.11. Given the recent results from [La1, La2], it appears likely that analogues of the results of the present section, and hence of Theorems 0.1–0.5, hold for *integral models* of Baily–Borel compactifications of arbitrary (pure) *PEL-type Shimura varieties* over primes of good reduction, and for Chow motives occurring as direct factors in the relative motive of integral models of Kuga families over such Shimura varieties. More precisely, the analogue of Lemma 8.1 holds in this context ([La2, Thm. 2.15 (1), Sect. 2C (1)], together with [La1, Thm. 6.4.1.1 2. and 3.]), and so does Lemma 8.2 (a) [La1, Thm. 7.2.4.1 4.]. As in the proof of Lemma 8.2, the strata of the model of the Baily–Borel compactification are quotients, denoted $[M_{\mathcal{H}}^{Z_{\mathcal{H}}}]$ in [La1, Thm. 7.2.4.1 4.], of smooth objects $M_{\mathcal{H}}^{Z_{\mathcal{H}}}$; given the limitations of our present understanding of [La1], it does not appear obvious that the quotient map $M_{\mathcal{H}}^{Z_{\mathcal{H}}} \to [M_{\mathcal{H}}^{Z_{\mathcal{H}}}]$ is finite and étale (which would imply the analogue of Lemma 8.2 (b)). Given what is called the "Hecke action" on the models of Baily–Borel compactifications in [La1, Prop. 7.2.5.1], the analogue of Lemma 8.3 would require the same information on $M_{\mathcal{H}}^{Z_{\mathcal{H}}} \to [M_{\mathcal{H}}^{Z_{\mathcal{H}}}]$; in addition, as in its proof, we would need $[\,\cdot h]$ to lift to a morphism $M_{\mathcal{H}}^{Z_{\mathcal{H}}} \to M_{\mathcal{H}'}^{Z_{\mathcal{H}'}}$, which is finite and étale. In order to obtain a statement analoguous to Lemma 8.4, one would need to analyze the restriction of the morphism of models of toroidal compactifications from [La2, Thm. 2.15 (2)] to individual strata. Finally, the proof of the analogue of Corollary 8.8 for integral models of arbitrary PEL-type Shimura varieties would involve [La2, Thm. 2.15 (4)].

Note that in the context of integral models of *Siegel varieties*, the missing geometric information is available [FC]; let us finish with a more detailed discussion of that situation. Fix an integer $g \ge 1$, consider the reductive group $G_{2g} := CSp_{2g,\mathbb{Q}}$, and the pure Shimura data $(G_{2g}, \mathfrak{H}_{2g})$ of [P1, Ex. 2.7]. Fix a neat open compact subgroup L of $G_{2g}(\mathbb{A}_f)$, and suppose that it lies between two *principal* open compact subgroups,

$$L_k \le L \le L_n \ , \ n \,|\, k \ ,$$

defined as the kernels of the reduction maps modulo k and n on $G_{2g}(\widehat{\mathbb{Z}})$, respectively. We also suppose that $n \ge 3$. It follows from the Main Theorem of Shimura–Taniyama on complex multiplication (see [D1, Thm. 4.19, Thm. 4.21]) that M^{L_k} equals the moduli space of principally polarized Abelian varieties of dimension g with principal level-k structure, denoted $\mathrm{A}_{g,k}$. Given the modular interpretation of the latter, it admits a canonical extension to $\mathbf{Spec}\,\mathbb{Z}[\frac{1}{k}]$. The resulting scheme is denoted by the same symbol $\mathrm{A}_{g,k}$; it is identical to the object from [FC, Def. I.4.4]. The scheme $\mathrm{A}_{g,k}$ is quasi-projective and smooth over $\mathbf{Spec}\,\mathbb{Z}[\frac{1}{k}]$, and the morphism $[\,\cdot 1] : M^{L_k} \longrightarrow\!\!\!\!\to M^{L_n}$ extends to a finite, étale morphism $\mathrm{A}_{g,k} \longrightarrow\!\!\!\!\to \mathrm{A}_{g,n} \otimes \mathbf{Spec}\,\mathbb{Z}[\frac{1}{k}]$ [FC, Rem. IV.6.2 (c)]. In fact [FC, Rem. IV.6.2 (d)], this morphism is identical to the map to the quotient by the natural (right) action of L_n/L_k on $\mathrm{A}_{g,k}$;

this action is free since $n \geq 3$. An integral model of M^L over $\mathbf{Spec}\,\mathbb{Z}[\frac{1}{k}]$ can thus be defined as the partial quotient by the action of L/L_k:

$$\mathrm{A}_{g,L} := \mathrm{A}_{g,k}/(L/L_k)\ .$$

It is quasi-projective and smooth over $\mathbf{Spec}\,\mathbb{Z}[\frac{1}{k}]$. This construction can be employed to show that *mutatis mutandis*, the main results from [FC] on $\mathrm{A}_{g,k}$ are equally valid for $\mathrm{A}_{g,L}$. In particular, the Baily-Borel compactification $(M^L)^*$ admits a proper model $\mathrm{A}^*_{g,L}$ over $\mathbf{Spec}\,\mathbb{Z}[\frac{1}{k}]$, defined as

$$\mathrm{A}^*_{g,L} := \mathrm{A}^*_{g,k}/(L/L_k)$$

[FC, Thm. V.2.5 (3)].

Now fix a second integer $s \geq 0$. The mixed Shimura data $(P, \mathfrak{X})$ is supposed to equal the s-fold fibre product of the data $(P_{2g}, \mathfrak{X}_{2g})$ of [P1, Ex. 2.25]; the underlying pure Shimura data thus equals $(G_{2g}, \mathfrak{H}_{2g})$, and the unipotent radical of P equals the s-th power of the standard representation V_{2g} of G_{2g}. Let K be an open compact sub-group of $P_{2g}(\mathbb{A}_f)$, whose image under the projection $\pi : (P, \mathfrak{X}) \to (G_{2g}, \mathfrak{H}_{2g})$ equals L, and which contains the image of L under the canonical Levi section of π. As in the pure case discussed above, the main results from [FC] on the s-fold fibre product $\mathrm{B}_{g,k} \to \mathrm{A}_{g,k}$ of the universal Abelian scheme over $\mathrm{A}_{g,k}$ (denoted Y in [loc. cit.]) generalize to the model $\pi : \mathrm{B}_{g,K} \to \mathrm{A}_{g,L}$ of M^K over $\mathbf{Spec}\,\mathbb{Z}[\frac{1}{k}]$. In particular, for suitable choices of complete cone decomposition $\mathfrak{S}$, the toroidal compactification $M^K(\mathfrak{S})$ extends to a proper, smooth model $\mathrm{B}_{g,K}(\mathfrak{S})$ over $\mathbf{Spec}\,\mathbb{Z}[\frac{1}{k}]$, and the morphism $\pi : M^K(\mathfrak{S}) \to (M^L)^*$ extends to a proper morphism $\mathrm{B}_{g,K}(\mathfrak{S}) \to \mathrm{A}^*_{g,L}$ [FC, Thm. VI.1.1], which we shall still denote by π.

Fix a direct factor N of $\pi_* \mathbb{1}_{\mathrm{B}_{g,K}}$, viewed as an object of the category $CHM(\mathrm{A}_{g,L})_F$; note that according to [O'S2, Prop. 5.1.1], the isomorphism classes of such N are in bijective correspondence with the direct factors of the generic fibre $\pi_* \mathbb{1}_{M^K}$ considered in the Introduction. Denote by $j : \mathrm{A}_{g,L} \hookrightarrow \mathrm{A}^*_{g,L}$ the open immersion, and by $i : \partial \mathrm{A}^*_{g,L} \hookrightarrow \mathrm{A}^*_{g,L}$ the complement of $\mathrm{A}_{g,L}$.

Theorem 8.12. *(a) There is a Chow motive $j_{!*} N \in CHM(\mathrm{A}^*_{g,L})_F$ extending N, and satisfying the following properties.*

(1) $j_{!*} N$ admits no non-zero direct factor belonging to $i_* CHM(\partial \mathrm{A}^*_{g,L})_F$.

(2) Any element of the kernel of

$$j^* : \mathrm{End}_{CHM(\mathrm{A}^*_{g,L})_F}(j_{!*} N) \longrightarrow \mathrm{End}_{CHM(\mathrm{A}_{g,L})_F}(N)$$

is nilpotent.

(b) The Chow motive $j_{!*} N$ satisfies analogues of properties (b)–(d) from Theorem 0.1.

(c) The construction of $j_{!*} N$ is compatible with Theorem 0.1 under pull-back

to the generic fibre $\mathbf{Spec}\,\mathbb{Q}$ *of* $\mathbf{Spec}\,\mathbb{Z}[\frac{1}{k}]$. *In particular, it is compatible with the Betti realization.*

Now let $[\ \cdot h] : M^L \to M^{L'}$ be a finite morphism associated to change of the "level" L, with

$$L_{k'} \leq L' \leq L_{n'}\ ,\ n' \,|\, k'$$

and $n' \geq 3$. Then $[\ \cdot h]$ extends to a finite morphism

$$\mathrm{A}^*_{g,L} \otimes \mathbf{Spec}\,\mathbb{Z}[\frac{1}{k \cdot k'}] \longrightarrow \mathrm{A}^*_{g,L'} \otimes \mathbf{Spec}\,\mathbb{Z}[\frac{1}{k \cdot k'}]\ ,$$

denoted by the same symbol $[\ \cdot h]$.

Theorem 8.13. *The intermediate extension is compatible with* $[\ \cdot h]_*$. *More precisely, the Chow motive* $[\ \cdot h]_*\, j_{!*}\, N \in CHM(\mathrm{A}^*_{g,L'} \otimes \mathbf{Spec}\,\mathbb{Z}[\frac{1}{k \cdot k'}])_F$ *satisfies the analogues of the properties from Theorem 8.12.*

Write m for the structure morphism $\mathrm{A}_{g,L} \to \mathbf{Spec}\,\mathbb{Z}[\frac{1}{k}]$. Fix $x \in G_{2g}(\mathbb{A}_f)$, and let k_x be an integer satisfying

$$L_{k_x} \leq L_x := L \cap x^{-1} L x\ .$$

Theorem 8.14. *The Hecke operator* LxL *acts on* $m_*\, j_{!*}\, N \otimes \mathbf{Spec}\,\mathbb{Z}[\frac{1}{k_x}]$, *in a way compatible with its action on* $m_* j_! N \otimes \mathbf{Spec}\,\mathbb{Z}[\frac{1}{k_x}]$ *and on* $m_* j_* N \otimes \mathbf{Spec}\,\mathbb{Z}[\frac{1}{k_x}]$, *and with the action from Theorem 0.5 on the generic fibre.*

Proof of Theorems 8.12, 8.13 and 8.14. Lemma 8.1 and Lemma 8.2 (a) hold, as was explained in Remark 8.11. As for Lemma 8.2 (b), the quotient map $M^{Z_\mathcal{H}}_\mathcal{H} \to [M^{Z_\mathcal{H}}_\mathcal{H}]$ (see Remark 8.11) is the identity [FC, Thm. V.2.5 (4)]; in particular, it is finite and étale. The same result (and the modular interpretation of $[\ \cdot h]$) shows that $[\ \cdot h]$ lifts to a morphism $M^{Z_\mathcal{H}}_\mathcal{H} \to M^{Z_{\mathcal{H}'}}_{\mathcal{H}'}$, which is finite and étale. We thus get Lemma 8.3, and also the extension of Hecke operators to integral models. As for Lemma 8.4, we refer to [FC, Thm. V.2.5 (5) and (6)] (if $s = 0$) and to [FC, Thm. VI.1.1 and its proof] (if $s \geq 1$)). **q.e.d.**

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